Coupling quantum matter to gravity: a systematic post-Newtonian approach

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CRC 1227
Designed Quantum States of Matter

RQI Circuit at ZARM, 24th November 2023

Outline

1. Motivation
2. Systematic post-Newtonian expansions
   - Simple two-particle atom
   - Massive spin-half particle
3. Conclusion

Review: D Giulini, A Großardt, PKS: *Coupling Quantum Matter and Gravity*, `arXiv:2207.05029`,
in: C Pfeifer, C Lämmerzahl (eds.), Modified and Quantum Gravity, Lect. Notes Phys. **1017** (Springer, Cham, 2023)

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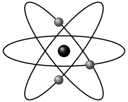

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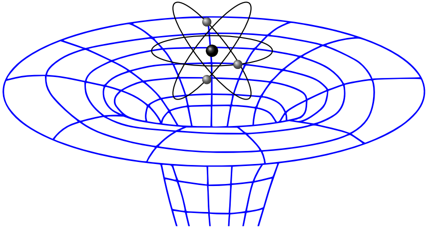

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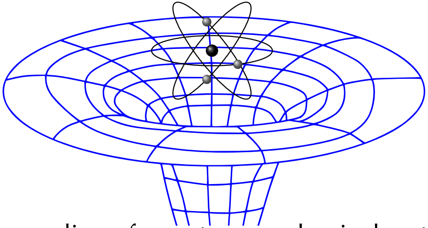

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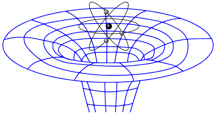

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- 'gravitational field': all ten components $g_{\mu\nu}$ of the metric

$$\begin{aligned} g_{00} &\Rightarrow \text{Newtonian potential} \rightsquigarrow \text{scalar part} \\ g_{0a} &\Rightarrow \text{gravitomagnetism} \rightsquigarrow \text{vector part} \\ g_{ab} &\Rightarrow \text{gravitational waves} \rightsquigarrow \text{tensor part} \end{aligned}$$

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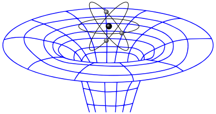

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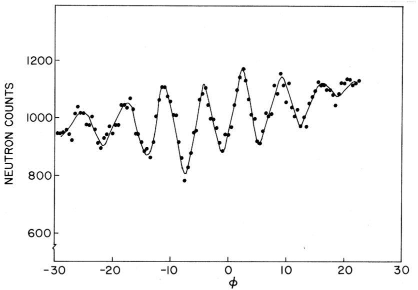
- Research in our group in Hannover (D Giulini, PKS) in this area:
  - Quantum systems in weak (post-Newtonian) gravitational fields
  - Systematic first-principles descriptions
  - 'Conservative' approach: based on established physics, using established language
  - Complete derivations that may then be applied for descriptions of experiments
    $\rightsquigarrow$ CRC 1227 DQ-mat

Post-Newtonian effects of ϕ

- Newtonian potential ϕ in quantum Hamiltonian:

$$H = \frac{\boldsymbol{P}^2}{2m} + m\phi$$

- First test: Colella, Overhauser, Werner 1975



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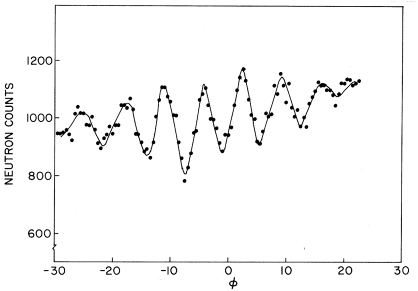

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- Higher-order coupling of ϕ?

Heuristic description of post-Newtonian effects

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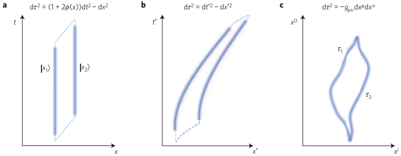
  - Similar: Interferometric measurement of special-relativistic time dilation

    S Loriani *et al.*: *Interference of Clocks: A Quantum Twin Paradox*, `arXiv:1905.09102`, Sci. Adv. **5**, eaax8966 (2019)

The need for systematic descriptions

- Heuristic descriptions need semi-classical notions such as worldlines



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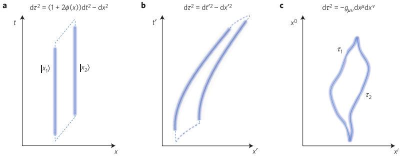

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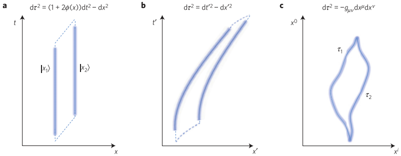

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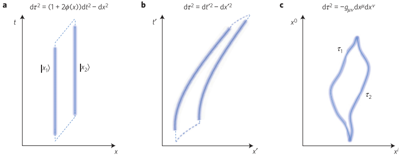

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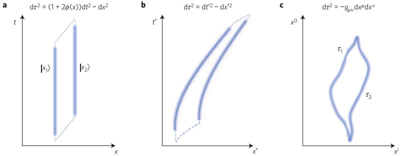

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- For reliable predictions: need systematic method, *complete* and *redundancy-free*

Systematic post-Newtonian expansions

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  - Weak gravitational fields
  - Approximately stationary spacetime ($\rightsquigarrow$ particles)
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- Background structures (defining space–time decomposition, small velocities, weak gravity): Minkowski metric η, inertial observer u
- In adapted coordinates $(x^0 = ct, x^a)$: start with full theory, expand fields as

$$g_{\mu\nu} = \eta_{\mu\nu} + \sum_{k=1}^{\infty} \varepsilon^k g_{\mu\nu}^{(k)} , \quad \dots \tag{2}$$

  in post-Newtonian parameter ε (often $\varepsilon = 1/c$)
- $\rightsquigarrow$ systematic perturbative inclusion of post-Newtonian effects!

Simple two-particle atom

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- Model composite system: Two oppositely charged point particles (without spin)
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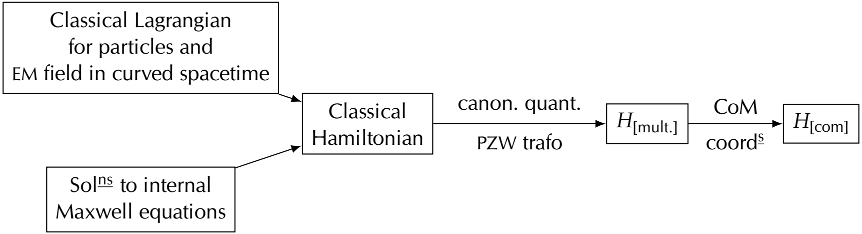

$$H_{[\mathrm{com}]} = H_{\mathrm{C}} + H_{\mathrm{A}} + H_{\mathrm{AL}} + H_{\mathrm{X}} \tag{3}$$

▸ Details of calculations

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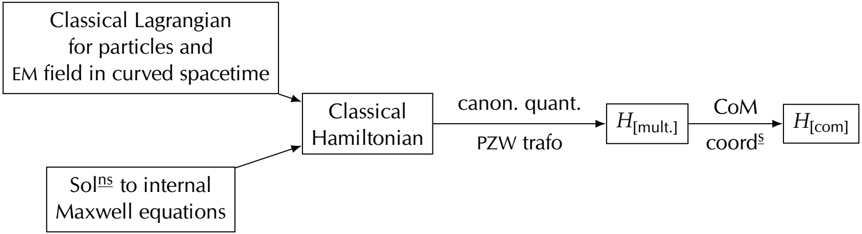

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Two-particle atom: resulting Hamiltonian

- Hamiltonian contains 'gravitational corrections'
- Simplified form in metric quantities

$$H_A \tag{4a}$$

$$H_C \tag{4b}$$

Full Hamiltonian

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$$H_{\mathrm{A}} = \frac{{}^{(3)}g_{\boldsymbol{R}}^{-1}(\boldsymbol{p}_r, \boldsymbol{p}_r)}{2\mu} + \frac{e_1 e_2}{4\pi\varepsilon_0 \sqrt{{}^{(3)}g_{\boldsymbol{R}}(\boldsymbol{r}, \boldsymbol{r})}} + c^{-2}(\text{SR \& 'Darwin' corrections} + \nabla\phi \text{ term}) + \mathrm{O}(c^{-4}) \quad \text{(4a)}$$

$$H_{\mathrm{C}} \quad \text{(4b)}$$

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$$H_{\mathrm{C}} = H_{\mathrm{point}}\left(\boldsymbol{P}, \boldsymbol{R}; M + \frac{H_{\mathrm{A}}}{c^2}\right) + \mathrm{O}(c^{-4}) \quad \text{(4b)}$$

- Effectively: CoM = point particle with mass $M + H_{\mathrm{int}}/c^2$

Full Hamiltonian

Massive spin-half particle

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Massive spin-half particle

- Spin-half field ψ, minimally coupled Dirac equation

$$(\mathrm{i}\gamma^\mu(\nabla_\mu - \mathrm{i}qA_\mu) - mc)\psi = 0 \tag{5}$$

- Restrict to one-particle sector $\rightsquigarrow$ effective description by positive-frequency *classical* solutions
- Systematic description from POV of observer on fixed worldline γ in two independent steps:
  1. *Weak gravity*: expansion in geodesic distance to γ
  2. *Slow velocities*: post-Newtonian expansion in c^{-1}
- Fully general situation: Spacetime can have *curvature* (R), observer can be *accelerated* ($\boldsymbol{a}$), may use *rotating* frame ($\boldsymbol{\omega}$)

▸ Details of calculations

A Alibabaei: master's thesis,
A Alibabaei, PKS, D Giulini: *Geometric post-Newtonian description of massive spin-half particles in curved spacetime*, arXiv:2307.04743, Class. Quantum Gravity **40**, 235014 (2023);
extending and correcting
T R Perche, J Neuser: *A wavefunction description for a localized quantum particle in curved spacetimes*, arXiv:2012.08539, Class. Quantum Gravity **38**, 175002 (2021)

Massive spin-half particle: resulting Hamiltonian

- End result: full post-Newtonian Pauli Hamiltonian describing dynamics from POV of observer on γ
(green: lowest-order terms with direct interpretation, red: mistakes in / extension of earlier literature)

$$\mathrm{i}\partial_\tau \tilde{\psi}_A = H_{\text{Pauli}} \tilde{\psi}_A \tag{6a}$$

$$\begin{aligned} H_{\text{Pauli}} = & -\frac{1}{2m}(\boldsymbol{\sigma} \cdot \boldsymbol{D})^2 - qcA_0 + m\boldsymbol{a} \cdot \boldsymbol{x} + \frac{mc^2}{2} R_{0l0m} x^l x^m - \frac{1}{2}\boldsymbol{\sigma} \cdot \boldsymbol{\omega} \\ & + \frac{\mathrm{i}c}{3} R_{0i} x^i + \frac{1}{8m} R + \frac{1}{4m} R_{00} - \frac{1}{8m^3c^2}(\boldsymbol{\sigma} \cdot \boldsymbol{D})^4 \\ & + \left\{ -\frac{1}{2mc^2} \boldsymbol{a} \cdot \boldsymbol{x} - \frac{1}{4m} R_{0l0m} x^l x^m \right\} (\boldsymbol{\sigma} \cdot \boldsymbol{D})^2 - \frac{1}{4m^2c^2} q \sigma^i \sigma^j D_i E_j \\ & + (\text{further } \boldsymbol{a}, R, \boldsymbol{\omega} \text{ corrections, incl. spin coupling}) + \mathrm{O}(c^{-3}) + \mathrm{O}(x^3) \end{aligned} \tag{6b}$$

▸ Full Hamiltonian

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Conclusion

- Quantum experiments under gravity require properly relativistic descriptions
- *Systematic*, therefore *exhaustive* and *redundancy-free* scheme for weak gravity: fully controlled post-Newtonian approximation
- Hamiltonian description of an atom in weak gravity:
  - First systematic and complete derivation up to order c^{-2}
  - Confirms intuitive point-particle picture: effectively $M \to M + H_{\text{int}}/c^2$
- Hamiltonian description of a slow spin-half particle in weak gravity:
  - Two steps: 1. weak gravity, 2. post-Newtonian
  - Systematic and complete derivation of post-Newtonian Pauli equation for *general* observer, using *general* frame
- Example application: atom interferometry $\to$ Klemens Hammerer's talk (now!)

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Many thanks for your attention!

Appendix: Details

Details of atomic calculation: setting the stage

- Physical spacetime metric: Eddington–Robertson PPN metric

$$g = \left(-1 \qquad\qquad\qquad \right) c^2 \mathrm{d}t^2 + \left(1 \qquad\qquad \right) \mathrm{d}\boldsymbol{x}^2 \tag{7}$$

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- Idea: perturbatively include gravity into calculations by S&B
  1. Couple ϕ to particles only
  2. Calculate EM Lagrangian with ϕ
  3. Repeat calculation of Hamiltonian including corrections to EM

Coupling of gravity to the particles

- Include coupling of ϕ to kinetic terms of particles:

$$\begin{aligned} L_{\text{point}} &= -mc^2\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu/c^2} + mc^2 \\ &= \frac{m\dot{\boldsymbol{x}}^2}{2}\left(1+\frac{\dot{\boldsymbol{x}}^2}{4c^2}\right) - \frac{2\gamma+1}{2}\,\frac{m\phi}{c^2}\dot{\boldsymbol{x}}^2 - m\phi\left(1+(2\beta-1)\frac{\phi}{2c^2}\right) + \mathrm{O}(c^{-4}) \end{aligned} \quad (8)$$

- Ignore coupling to EM
- Repeating calculation by S&B:

$$H_{\text{C,new}} = H_{\text{C,non-grav.}} + \frac{2\gamma+1}{2Mc^2}\boldsymbol{P}\cdot\phi(\boldsymbol{R})\boldsymbol{P} + \left(M+\frac{\boldsymbol{p}_r^2}{2\mu c^2}\right)\phi(\boldsymbol{R}) + (2\beta-1)\frac{M\phi(\boldsymbol{R})^2}{2c^2} \quad (9a)$$

$$H_{\text{A,new}} = H_{\text{A,non-grav.}} + 2\gamma\frac{\phi(\boldsymbol{R})}{c^2}\frac{\boldsymbol{p}_r^2}{2\mu} - \frac{2\gamma+1}{2c^2}\,\frac{m_1-m_2}{m_1 m_2}\boldsymbol{p}_r\cdot(\boldsymbol{r}\cdot\nabla\phi(\boldsymbol{R}))\boldsymbol{p}_r \quad (9b)$$

Coupling of gravity to the EM field

- Start from EM action in gravity
- Rewrite Maxwell equations in gravity in terms of 'flat' equations
- Solve perturbatively
- 'Internal' potentials: $\boldsymbol{\mathcal{A}}^{\perp} = \boldsymbol{\mathcal{A}}^{\perp}_{\text{non-grav.}} + O(c^{-4})$,

$$\begin{aligned}\phi_{\text{el.}}(\boldsymbol{x}, t) = \phi_{\text{el.,non-grav.}}(\boldsymbol{x}, t) \\ + c^{-2} \bigg[\frac{\gamma + 1}{4\pi\varepsilon_0} \int \mathrm{d}^3x' \frac{\phi(\boldsymbol{x}', t)\rho(\boldsymbol{x}', t)}{|\boldsymbol{x} - \boldsymbol{x}'|} \\ - \frac{\gamma + 1}{4\pi} \int \mathrm{d}^3x' \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} (\nabla\phi \cdot \nabla\phi_{\text{el.,non-grav.}})(\boldsymbol{x}', t) \bigg] \\ + O(c^{-4})\end{aligned} \tag{10}$$

- $\rightsquigarrow$ EM Lagrangian with gravitational corrections

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The full atomic Hamiltonian

$$H_{[\mathrm{com}]} = H_{\mathrm{C}} + H_{\mathrm{A}} + H_{\mathrm{AL}} + H_{\mathrm{X}} + H_{\mathrm{deriv,new}} + \mathrm{O}(c^{-4}) \tag{11a}$$

$$H_{\mathrm{C}} = \frac{\boldsymbol{P}^2}{2M}\left[1 - \frac{1}{Mc^2}\left(\frac{\boldsymbol{p}_r^2}{2\mu} - \frac{e^2}{4\pi\varepsilon_0 r}\right)\right] + \left[M + \frac{1}{c^2}\left(\frac{\boldsymbol{p}_r^2}{2\mu} - \frac{e^2}{4\pi\varepsilon_0 r}\right)\right]\phi(\boldsymbol{R}) - \frac{\boldsymbol{P}^4}{8M^3c^2} + \frac{2\gamma+1}{2Mc^2}\boldsymbol{P}\cdot\phi(\boldsymbol{R})\boldsymbol{P} + (2\beta-1)\frac{M\phi(\boldsymbol{R})^2}{2c^2} \tag{11b}$$

$$H_{\mathrm{A}} = \left(1 + 2\gamma\frac{\phi(\boldsymbol{R})}{c^2}\right)\frac{\boldsymbol{p}_r^2}{2\mu} - \left(1 + \gamma\frac{\phi(\boldsymbol{R})}{c^2}\right)\frac{e^2}{4\pi\varepsilon_0 r} - \frac{m_1^3 + m_2^3}{M^3}\frac{\boldsymbol{p}_r^4}{8\mu^3c^2} - \frac{e^2}{4\pi\varepsilon_0}\frac{1}{2\mu Mc^2}\left(\boldsymbol{p}_r\cdot\frac{1}{r}\boldsymbol{p}_r + \boldsymbol{p}_r\cdot\boldsymbol{r}\frac{1}{r^3}\boldsymbol{r}\cdot\boldsymbol{p}_r\right)$$
$$- \frac{2\gamma+1}{2c^2}\frac{m_1 - m_2}{m_1 m_2}\boldsymbol{p}_r\cdot(\boldsymbol{r}\cdot\boldsymbol{\nabla}\phi(\boldsymbol{R}))\boldsymbol{p}_r - \frac{\gamma+1}{c^2}\frac{e^2}{8\pi\varepsilon_0 r}\frac{m_2 - m_1}{M}\boldsymbol{r}\cdot\boldsymbol{\nabla}\phi(\boldsymbol{R}) \tag{11c}$$

$$H_{\mathrm{AL}} = -\boldsymbol{E}^{\perp}(\boldsymbol{R})\cdot\boldsymbol{d} + \frac{1}{2M}\{\boldsymbol{P}\cdot[\boldsymbol{d}\times\boldsymbol{B}(\boldsymbol{R})] + \mathrm{H.c.}\} - \frac{m_1 - m_2}{4m_1 m_2}\{\boldsymbol{p}_r\cdot[\boldsymbol{d}\times\boldsymbol{B}(\boldsymbol{R})] + \mathrm{H.c.}\}$$
$$+ \frac{1}{8\mu}(\boldsymbol{d}\times\boldsymbol{B}(\boldsymbol{R}))^2 + \varepsilon_0\int \mathrm{d}^3\boldsymbol{x}\,(\gamma+1)\,\phi_{\mathrm{el.}}^{(0)}\frac{\boldsymbol{\nabla}\phi}{c^2}\cdot\boldsymbol{E}^{\perp} \tag{11d}$$

$$H_{\mathrm{X}} = -\frac{(\boldsymbol{P}\cdot\boldsymbol{p}_r)^2}{2M^2\mu c^2} + \frac{e^2}{4\pi\varepsilon_0 r}\frac{(\boldsymbol{P}\cdot\boldsymbol{r}/r)^2}{2M^2c^2} + \frac{m_1 - m_2}{2\mu M^2c^2}\left\{(\boldsymbol{P}\cdot\boldsymbol{p}_r)\boldsymbol{p}_r^2/\mu - \frac{e^2}{8\pi\varepsilon_0}\left[\frac{1}{r}\boldsymbol{P}\cdot\boldsymbol{p}_r + \frac{1}{r^3}(\boldsymbol{P}\cdot\boldsymbol{r})(\boldsymbol{r}\cdot\boldsymbol{p}_r) + \mathrm{H.c.}\right]\right\} \tag{11e}$$

$$H_{\mathrm{deriv,new}} = \frac{2\gamma+1}{2Mc^2}[\boldsymbol{P}\cdot(\boldsymbol{r}\cdot\boldsymbol{\nabla}\phi(\boldsymbol{R}))\boldsymbol{p}_r + \mathrm{H.c.}] \tag{11f}$$

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Spinor calculation, step 1: 'Weak-gravity' expansion in generalised Fermi normal coordinates

Idea

Fermi normal coordinates = 'proper coordinates' for observer along worldline γ

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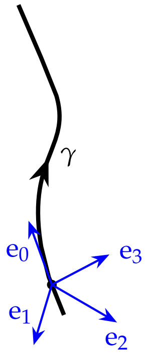
Fermi normal coordinates = 'proper coordinates' for observer along worldline γ

- Arbitrary orthonormal vector fields (e_i) along γ



Spinor calculation, step 1: 'Weak-gravity' expansion in generalised Fermi normal coordinates

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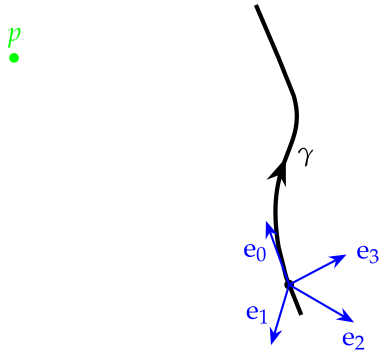
- Arbitrary orthonormal vector fields (e_i) along γ
- Point p close to γ: unique spacelike geodesic to γ



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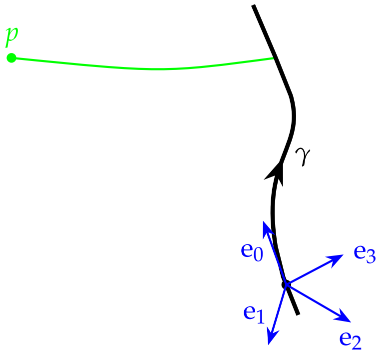

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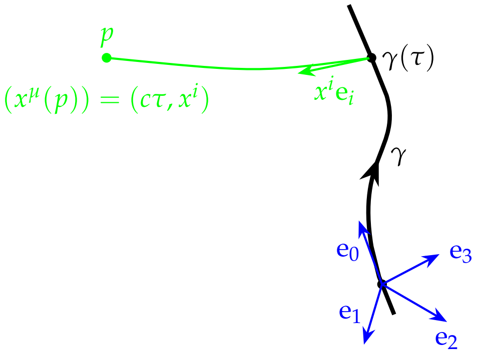

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Fermi normal coordinates = 'proper coordinates' for observer along worldline γ

- Arbitrary orthonormal vector fields (e_i) along γ
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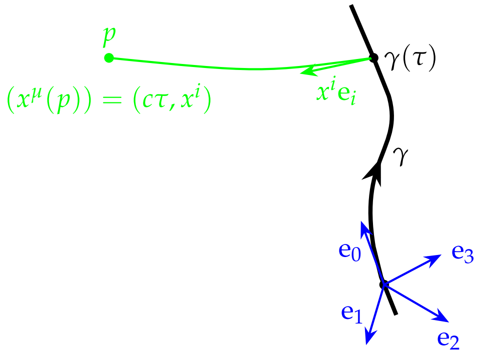
- Coordinates for p: proper time of starting point and initial direction of this geodesic
- Expand Dirac equation in geodesic distance: weak gravity & weak inertial effects



$$R_{IJKL} \cdot \|\boldsymbol{x}\|^2 \ll 1, \qquad \frac{R_{IJKL;M}}{R_{NOPQ}} \cdot \|\boldsymbol{x}\| \ll 1, \qquad \frac{\boldsymbol{a}}{c^2} \cdot \boldsymbol{x} \ll 1, \qquad \frac{\boldsymbol{\omega}}{c} \cdot \boldsymbol{x} \ll 1 \tag{12}$$

Spinor calculation, step 2: Post-Newtonian expansion

- Post-Newtonian expansion of Dirac field:

$$\psi = \mathrm{e}^{\mathrm{i}c^2 S}\tilde{\psi} \text{ with } S = \mathrm{O}(c^0), \tilde{\psi} = \sum_{k=0}^{\infty} c^{-k}\tilde{\psi}^{(k)} \tag{13}$$

- Insert into weak-gravity Dirac equation, evaluate order by order
- Leading order $c^3 \implies \partial_i S = 0$
- Order c^2: $-(\partial_\tau S)\tilde{\psi}^{(0)} = \gamma^0 m\tilde{\psi}^{(0)} \implies \partial_\tau S = \pm m \rightsquigarrow S = -m\tau$
- Now decompose (in Dirac representation) $\tilde{\psi}^{(k)} = \begin{pmatrix} \tilde{\psi}_A^{(k)} \\ \tilde{\psi}_B^{(k)} \end{pmatrix}$, with $\tilde{\psi}_B^{(0)} = 0$
- Higher orders: equations for $\tilde{\psi}_{A,B}^{(k)}$
- Eliminate $\tilde{\psi}_B \rightsquigarrow$ complete post-Newtonian Pauli equation for $\tilde{\psi}_A$

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The full Pauli Hamiltonian

$$
\begin{aligned}
H_{\text{Pauli}} = {} & \Big\{ -\frac{1}{2m} - \frac{1}{2mc^2}\boldsymbol{a}\cdot\boldsymbol{x} - \frac{1}{4m}R_{0l0m}x^l x^m - \frac{1}{8m}R_{0l0m;n}x^l x^m x^n - \frac{1}{24m}R_{0k0l;mn}x^k x^l x^m x^n \Big\}(\boldsymbol{\sigma}\cdot\boldsymbol{D})^2 - \frac{1}{8m^3c^2}(\boldsymbol{\sigma}\cdot\boldsymbol{D})^4 \\
& + \Big\{ -\frac{1}{6m}R^{i}{}_{l}{}^{j}{}_{m}x^l x^m - \frac{1}{12m}R^{i}{}_{l}{}^{j}{}_{m;n}x^l x^m x^n - \frac{1}{40m}R^{i}{}_{k}{}^{j}{}_{l;mn}x^k x^l x^m x^n \Big\} D_i D_j + \Big\{ \mathrm{i}(\boldsymbol{\omega}\times\boldsymbol{x})^j - \frac{2\mathrm{i}c}{3}R_{0l}{}^{j}{}_{m}x^l x^m \\
& - \frac{\mathrm{i}c}{4}R_{0l}{}^{j}{}_{m;n}x^l x^m x^n - \frac{1}{4mc^2}a^j - \frac{\mathrm{i}}{4mc^2}(\boldsymbol{\sigma}\times\boldsymbol{a})^j + \frac{1}{12m}(4R^{j}{}_{l} + R_{0}{}^{j}{}_{0l})x^l + \frac{\mathrm{i}}{8m}\sigma^k(-2\varepsilon^{ij}{}_{k}R_{0l0i} + \varepsilon^{im}{}_{k}R^{j}{}_{lim})x^l \\
& + \frac{1}{24m}\Big(5R^{j}{}_{l;m} - 3R_{0}{}^{j}{}_{0l;m} - R_{0l0m}{}^{;j} - R^{j}{}_{l}{}^{i}{}_{m;i} - \mathrm{i}\varepsilon^{ij}{}_{k}\sigma^k(2R_{0i0l;m} + R_{0l0m;i}) + 2\mathrm{i}\varepsilon^{in}{}_{k}\sigma^k R^{j}{}_{lin;m}\Big)x^l x^m \\
& + \frac{1}{120m}\Big(9R^{j}{}_{l;mn} - 6R_{0}{}^{j}{}_{0l;mn} - 5R_{0l0m}{}^{;j}{}_{n} - 3R^{j}{}_{l}{}^{i}{}_{m;in}\Big)x^l x^m x^n + \frac{\mathrm{i}}{96m}\sigma^k\Big(-4\varepsilon^{ij}{}_{k}(R_{0i0l;mn} + R_{0l0m;ni}) + 3\varepsilon^{ir}{}_{k}R^{j}{}_{lir;mn}\Big)x^l x^m x^n \Big\} D_j \\
& - qA_\tau + m\boldsymbol{a}\cdot\boldsymbol{x} + \frac{mc^2}{2}R_{0l0m}x^l x^m - \frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\omega} + \frac{\mathrm{i}c}{3}R_{0l}x^l - \frac{c}{4}\varepsilon^{ij}{}_{k}\sigma^k R_{0lij}x^l + \frac{\mathrm{i}c}{24}(5R_{0l;m} - R_{0l}{}^{i}{}_{m;i})x^l x^m - \frac{c}{8}\varepsilon^{ij}{}_{k}\sigma^k R_{0lij;m}x^l x^m \\
& + \frac{1}{8m}R + \frac{1}{4m}R_{00} + \frac{1}{16m}(R_{;l} + 2R^{i}{}_{l;i})x^l + \frac{\mathrm{i}}{24m}\varepsilon^{ij}{}_{k}\sigma^k(R_{0i0l;j} - 2R_{il;j})x^l + \frac{1}{48m}\Big(R_{;lm} + 4R^{i}{}_{l;im} + \mathrm{i}\varepsilon^{ij}{}_{k}\sigma^k(R_{0i0l;jm} - 3R_{il;jm})\Big)x^l x^m \\
& - \frac{q}{4m^2c^2}\sigma^i\sigma^j D_i E_j - \frac{q}{12m}(R_{lm} + R_{0l0m})x^l x^m \boldsymbol{\sigma}\cdot\boldsymbol{B} + \frac{q}{12m}\sigma^j R_{iljm}x^l x^m B^i + \frac{\mathrm{i}q}{4m^2c^2}\boldsymbol{\sigma}\cdot(\boldsymbol{\omega}\times\boldsymbol{B}) + \frac{q}{2m^2c^2}\boldsymbol{\omega}\cdot\boldsymbol{B} \\
& + \frac{q}{4m^2c^2}(\omega_j x^i - \omega^i x_j)D_i B^j + \frac{\mathrm{i}q}{4m^2c^2}(\boldsymbol{\sigma}\cdot(\boldsymbol{\omega}\times\boldsymbol{x}))\boldsymbol{B}\cdot\boldsymbol{D} - \frac{\mathrm{i}q}{4m^2c^2}\sigma^j(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\boldsymbol{D}B_j + O(c^{-3}) + O(x^3)
\end{aligned}
\tag{14}
$$

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