Coupling quantum matter to gravity: a systematic post-Newtonian approach

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Designed Quantum States of Matter

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Outline

Review: D Giulini, A Großardt, PKS: *Coupling Quantum Matter and Gravity*, `arXiv:2207.05029`

The problem

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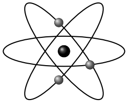
General motivation



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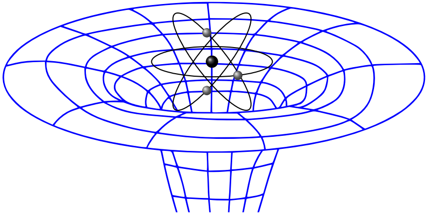

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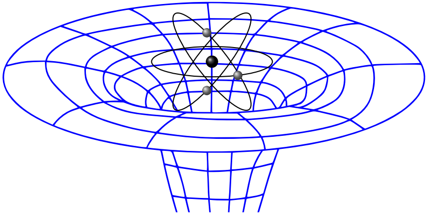

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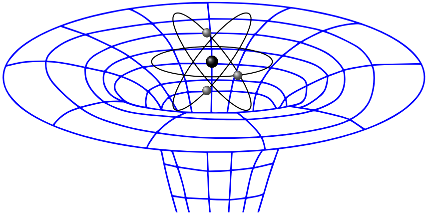

- Wish to understand the coupling of quantum-mechanical systems to an external gravitational field
- 'gravitational field': all ten components $g_{\mu\nu}$ of the metric

$$\begin{aligned} g_{00} &\Rightarrow \text{Newtonian potential} \rightsquigarrow \text{scalar part} \\ g_{0a} &\Rightarrow \text{gravitomagnetism} \rightsquigarrow \text{vector part} \\ g_{ab} &\Rightarrow \text{gravitational waves} \rightsquigarrow \text{tensor part} \end{aligned}$$

Post-Newtonian effects of ϕ

- Newtonian potential ϕ in quantum Hamiltonian:

$$H = \frac{\boldsymbol{P}^2}{2m} + m\phi$$

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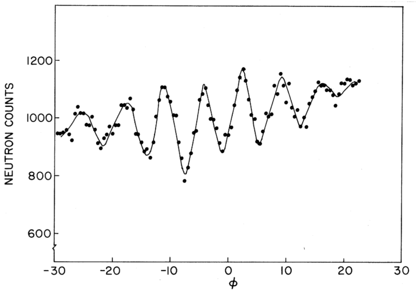

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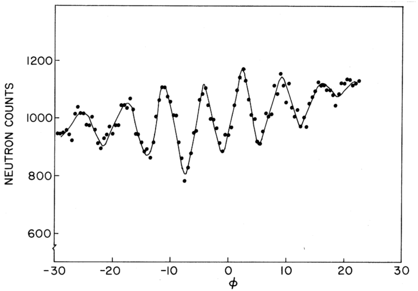

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- Higher-order coupling of ϕ?

Heuristic description of post-Newtonian effects

- Start with Newtonian description

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  - Similar: Interferometric measurement of special-relativistic time dilation

    S Loriani *et al.*: *Interference of Clocks: A Quantum Twin Paradox*, `arXiv:1905.09102`, Sci. Adv. **5**, eaax8966 (2019)

Conceptual issues of available descriptions

- No guarantee of completeness or independence of 'relativistic effects':
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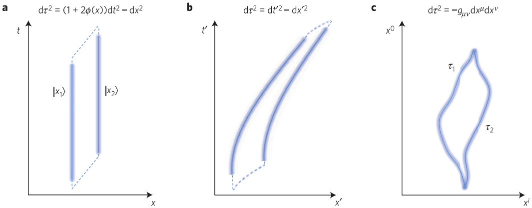

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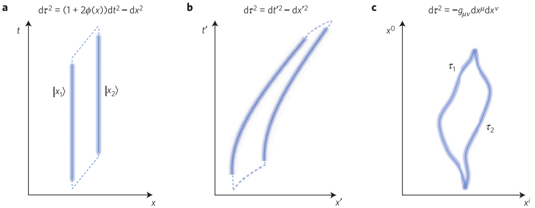
- Need semi-classical notions such as wordlines
  - Assumption of separating state $|\psi\rangle_{\text{tot}} = |\psi\rangle_{\text{ext}} \otimes |\psi\rangle_{\text{int}}$ even though interactions are the point of interest
  - Restriction to semi-classical central states



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The need for systematic descriptions

- For reliable predictions: need systematic method, *complete* and *exhaustive*
- Proper *derivation* of couplings, starting from well-established first principles
- No *a priori* restrictions on the state of matter
- More fundamental understanding, and the only way to properly test predictions

Framework: post-Newtonian expansions

Review: D Giulini, A Großardt, PKS: *Coupling Quantum Matter and Gravity*, `arXiv:2207.05029`

Why post-Newtonian expansions?

- Special-relativistic theories of matter: Equivalence principle $\rightsquigarrow$ minimal coupling: '$\eta_{\mu\nu} \to g_{\mu\nu}, \partial_\mu \to \nabla_\mu$'
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  - Weak gravitational fields
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- Include post-Newtonian effects perturbatively

‘Weak gravity’ – geometric setting

- ‘Post-Newtonian expansion’ needs notions of:
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- Background structures: Minkowski metric η, inertial observer u
- Adapted coordinates $(x^0 = ct, x^a)$: $u = \partial/\partial t$, $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$

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- Background structures: Minkowski metric η, inertial observer u
- Adapted coordinates $(x^0 = ct, x^a)$: $u = \partial/\partial t$, $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$
- Physical spacetime metric: power series in c^{-1}

$$g_{\mu\nu} = \eta_{\mu\nu} + \sum_{k=1}^{\infty} c^{-k} g^{(k)}_{\mu\nu} \tag{2}$$

$\rightsquigarrow$ systematic expansion of theory in c^{-1}!

Model systems

1. The problem
2. Framework: post-Newtonian expansions
3. Model systems
   - Simple two-particle atom
   - Massive spin-half particle
4. Conclusion

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Simple two-particle atom

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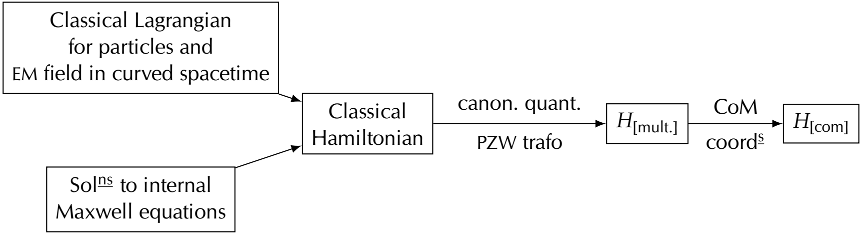
Simple two-particle atom

- Model composite system: Two oppositely charged point particles (without spin)
- Sonnleitner and Barnett 2018: Systematic derivation of 'approximately relativistic' Hamiltonian in external EM field, i.e. complete to $O(c^{-2})$

  M Sonnleitner, S M Barnett: *Mass-energy and anomalous friction in quantum optics*, arXiv:1806.00234, PRA **98**, 042106 (2018)
- Our work: extension to weak gravitational field (Eddington–Robertson PPN metric)

PKS, D Giulini: *Post-Newtonian Hamiltonian description of an atom in a weak gravitational field*, arXiv:1908.06929, Phys. Rev. A **100**, 052116 (2019);
extended in
PKS: *Post-Newtonian Description of Quantum Systems in Gravitational Fields*, arXiv:2009.11319, doctoral thesis, https://doi.org/10.15488/10085

Two-particle atom: calculational scheme



$$H_{[\mathrm{com}]} = H_{\mathrm{C}} + H_{\mathrm{A}} + H_{\mathrm{AL}} + H_{\mathrm{X}} \qquad (3)$$

▸ Details of calculations

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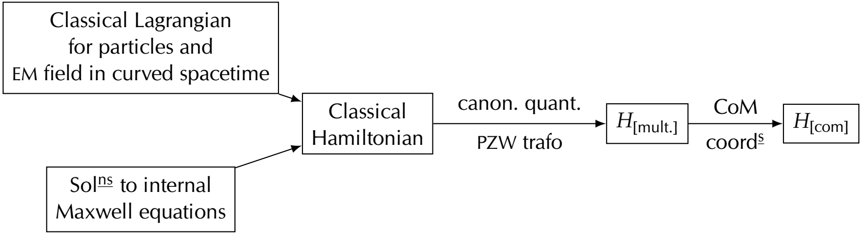

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Two-particle atom: resulting Hamiltonian

- Hamiltonian contains 'gravitational corrections'
- Simplified form in metric quantities

$$H_{\mathrm{A,final}} \tag{4a}$$

$$H_{\mathrm{C,final}} \tag{4b}$$

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$$H_{\mathrm{A,final}} = \frac{{}^{(3)}g_R^{-1}(\boldsymbol{p_r},\boldsymbol{p_r})}{2\mu} + \frac{e_1 e_2}{4\pi\varepsilon_0 \sqrt{{}^{(3)}g_R(\boldsymbol{r},\boldsymbol{r})}} + (\text{SR \& 'Darwin' corrections} + \boldsymbol{\nabla}\phi \text{ term}) \tag{4a}$$

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$$H_{\mathrm{C,final}} = H_{\mathrm{point}}\left(\boldsymbol{P}, \boldsymbol{R}; M + \frac{H_{\mathrm{A,final}}}{c^2}\right) \tag{4b}$$

▸ Full Hamiltonian

Massive spin-half particle

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Massive spin-half particle

- Spin-half field ψ, minimally coupled Dirac equation

$$(\mathrm{i}\gamma^\mu(\nabla_\mu - \mathrm{i}qA_\mu) - mc)\psi = 0 \tag{5}$$

- Restrict to one-particle sector $\rightsquigarrow$ effective description by positive-frequency *classical* solutions
- Systematic description from POV of observer on fixed worldline γ in two independent steps:
  1. *Weak gravity*: expansion in geodesic distance to γ
  2. *Slow velocities*: post-Newtonian expansion in c^{-1}
- Fully general situation: Spacetime can have *curvature* (R), observer can be *accelerated* ($\boldsymbol{a}$), may use *rotating* frame ($\boldsymbol{\omega}$)

A Alibabaei: *Geometric post-Newtonian description of spin-half particles in curved spacetime*, `arXiv:2204.05997`, master's thesis;
A Alibabaei, PKS, D Giulini: in preparation

Step 1: 'Weak-gravity' expansion in generalised Fermi normal coordinates

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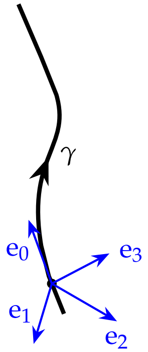

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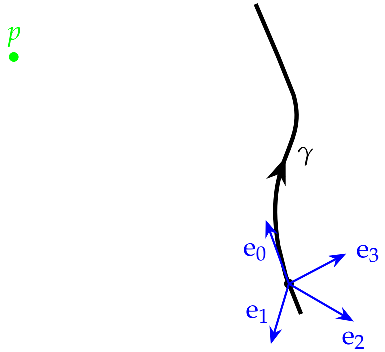

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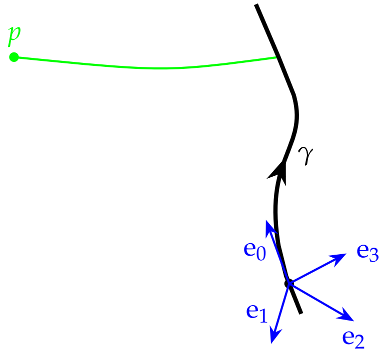

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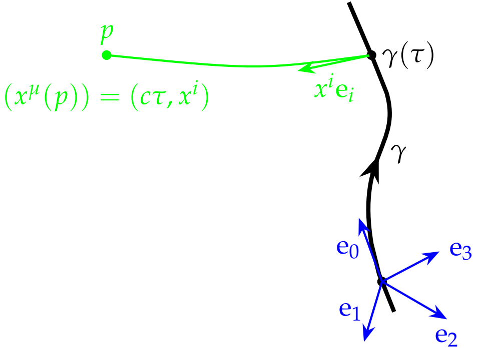

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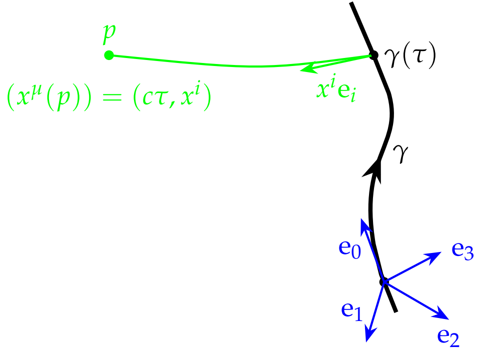

- Arbitrary orthonormal vector fields (e_i) along γ
- Point p close to γ: unique spacelike geodesic to γ
- Coordinates for p: proper time of starting point and initial direction of this geodesic
- Expand Dirac equation in geodesic distance: weak gravity & weak inertial effects

$$R_{IJKL} \cdot \boldsymbol{x}^2 \ll 1, \qquad \frac{\boldsymbol{a}}{c^2} \cdot \boldsymbol{x} \ll 1, \qquad \frac{\boldsymbol{\omega}}{c} \cdot \boldsymbol{x} \ll 1, \tag{6}$$

Step 2: Post-Newtonian expansion

- Post-Newtonian expansion of Dirac field:

$$\psi = \mathrm{e}^{-\mathrm{i}mc^2\tau} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}, \qquad \psi_{A,B} = \psi^{(0)}_{A,B} + c^{-1}\psi^{(1)}_{A,B} + c^{-2}\psi^{(2)}_{A,B} + \mathrm{O}(c^{-3}) \tag{7}$$

- Insert into weak-gravity Dirac equation ⇝ complete post-Newtonian Pauli equation (red: mistakes in literature)

$$\mathrm{i}\partial_\tau \psi_A = H_{\mathrm{Pauli}} \psi_A \tag{8a}$$

$$\begin{aligned} H_{\mathrm{Pauli}} = & -\frac{1}{2m}(\boldsymbol{\sigma}\cdot\boldsymbol{D})^2 - qcA_0 + m(\boldsymbol{a}\cdot\boldsymbol{x}) + \frac{mc^2}{2}R_{0l0m}x^l x^m \\ & + \frac{1}{8m}R + \frac{1}{4m}R_{00} + \frac{\mathrm{i}c}{3}R_{0i}x^i - \frac{1}{2}\sigma_i\omega^i - \frac{1}{8m^3c^2}(\boldsymbol{\sigma}\cdot\boldsymbol{D})^4 \\ & + \left\{ -\frac{1}{2mc^2}(\boldsymbol{a}\cdot\boldsymbol{x}) - \frac{1}{4m}R_{0l0m}x^l x^m \right\}(\boldsymbol{\sigma}\cdot\boldsymbol{D})^2 - \frac{1}{4m^2c^2}q\sigma^i\sigma^j D_i E_j \\ & + (\text{further } \boldsymbol{a}, R, \boldsymbol{\omega} \text{ corrections, incl. spin coupling}) + \mathrm{O}(c^{-3}) + \mathrm{O}(x^3) \end{aligned} \tag{8b}$$

▸ Full Hamiltonian

Conclusion

- Quantum experiments under gravity require properly relativistic descriptions
- *Systematic* and *exhaustive* scheme: fully controlled post-Newtonian approximation
- Hamiltonian description of an atom in weak gravity:
  - First systematic and complete derivation up to order c^{-2}
  - Confirms intuitive point-particle picture: effectively $M \to M + H_{\text{int}}/c^2$
- Hamiltonian description of a slow spin-half particle in weak gravity:
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Many thanks for your attention!

Appendix: Details

Details of atomic calculation: setting the stage

- Physical spacetime metric: Eddington–Robertson PPN metric

$$g = \Big(-1 \qquad\qquad\qquad\qquad \Big) c^2 \mathrm{d}t^2 + \Big(1 \qquad\qquad \Big) \mathrm{d}\boldsymbol{x}^2 \tag{9}$$

Details of atomic calculation: setting the stage

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- Physical spacetime metric: Eddington–Robertson PPN metric

$$g = \left(-1 - 2\frac{\phi}{c^2} - 2\ \frac{\phi^2}{c^4}\right) c^2 \mathrm{d}t^2 + \left(1 - 2\ \frac{\phi}{c^2}\right) \mathrm{d}\boldsymbol{x}^2 \tag{9}$$

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- Physical spacetime metric: Eddington–Robertson PPN metric; GR: $\beta = \gamma = 1$

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- Idea: perturbatively include gravity into calculations by S&B
  1. Couple ϕ to particles only
  2. Calculate EM Lagrangian with ϕ
  3. Repeat calculation of Hamiltonian including corrections to EM

Coupling of gravity to the particles

- Include coupling of ϕ to kinetic terms of particles:

$$
\begin{aligned}
L_{\text{point}} &= -mc^2\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu/c^2} + mc^2 \\
&= \frac{m\dot{\boldsymbol{x}}^2}{2}\left(1 + \frac{\dot{\boldsymbol{x}}^2}{4c^2}\right) - \frac{2\gamma+1}{2}\frac{m\phi}{c^2}\dot{\boldsymbol{x}}^2 \\
&\quad - m\phi\left(1 + (2\beta - 1)\frac{\phi}{2c^2}\right) + \mathrm{O}(c^{-4})
\end{aligned} \tag{10}
$$

- Ignore coupling to EM
- Repeating calculation by S&B:

$$
H_{\text{C,new}} = H_{\text{C}} + \frac{2\gamma+1}{2Mc^2}\boldsymbol{P}\cdot\phi(\boldsymbol{R})\boldsymbol{P} + \left(M + \frac{\boldsymbol{p}_r^2}{2\mu c^2}\right)\phi(\boldsymbol{R}) + (2\beta - 1)\frac{M\phi(\boldsymbol{R})^2}{2c^2} \tag{11a}
$$

$$
H_{\text{A,new}} = H_{\text{A}} + 2\gamma\frac{\phi(\boldsymbol{R})}{c^2}\frac{\boldsymbol{p}_r^2}{2\mu} - \frac{2\gamma+1}{2c^2}\frac{m_1 - m_2}{m_1 m_2}\boldsymbol{p}_r\cdot(\boldsymbol{r}\cdot\boldsymbol{\nabla}\phi(\boldsymbol{R}))\boldsymbol{p}_r \tag{11b}
$$

Coupling of gravity to the EM field

- Start from EM action in gravity
- Rewrite Maxwell equations in gravity in terms of 'flat' equations
- Solve perturbatively
- 'Internal' potentials: $\boldsymbol{\mathcal{A}}^{\perp} = \boldsymbol{\mathcal{A}}^{\perp}_{\text{non-grav.}} + \mathrm{O}(c^{-4})$,

$$\begin{aligned}\phi_{\text{el.}}(\boldsymbol{x},t) &= \phi_{\text{el.,non-grav.}}(\boldsymbol{x},t) \\ &+ c^{-2}\left[\frac{\gamma+1}{4\pi\varepsilon_0}\int \mathrm{d}^3x' \frac{\phi(\boldsymbol{x}',t)\rho(\boldsymbol{x}',t)}{|\boldsymbol{x}-\boldsymbol{x}'|}\right. \\ &\quad \left. - \frac{\gamma+1}{4\pi}\int \mathrm{d}^3x' \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}(\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi_{\text{el.,non-grav.}})(\boldsymbol{x}',t)\right] \\ &+ \mathrm{O}(c^{-4}) \qquad (12)\end{aligned}$$

- $\rightsquigarrow$ EM Lagrangian with gravitational corrections

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The full atomic Hamiltonian

$$
\begin{aligned}
H_{[\mathrm{com}],\mathrm{final}} &= H_{\mathrm{C,final}} + H_{\mathrm{A,final}} + H_{\mathrm{AL,final}} + H_{\mathrm{L,final}} + H_{\mathrm{X}} + H_{\mathrm{deriv,new}} + \mathrm{O}(c^{-4}) \\
H_{\mathrm{C,final}} &= \frac{\boldsymbol{P}^2}{2M}\left[1 - \frac{1}{Mc^2}\left(\frac{\boldsymbol{p}_r^2}{2\mu} - \frac{e^2}{4\pi\varepsilon_0 r}\right)\right] + \left[M + \frac{1}{c^2}\left(\frac{\boldsymbol{p}_r^2}{2\mu} - \frac{e^2}{4\pi\varepsilon_0 r}\right)\right]\phi(\boldsymbol{R}) \\
&\quad - \frac{\boldsymbol{P}^4}{8M^3c^2} + \frac{2\gamma+1}{2Mc^2}\boldsymbol{P}\cdot\phi(\boldsymbol{R})\boldsymbol{P} + (2\beta-1)\frac{M\phi(\boldsymbol{R})^2}{2c^2} \\
H_{\mathrm{A,final}} &= \left(1 + 2\gamma\frac{\phi(\boldsymbol{R})}{c^2}\right)\frac{\boldsymbol{p}_r^2}{2\mu} - \left(1 + \gamma\frac{\phi(\boldsymbol{R})}{c^2}\right)\frac{e^2}{4\pi\varepsilon_0 r} \\
&\quad - \frac{m_1^3 + m_2^3}{M^3}\frac{\boldsymbol{p}_r^4}{8\mu^3c^2} - \frac{e^2}{4\pi\varepsilon_0}\frac{1}{2\mu Mc^2}\left(\boldsymbol{p}_r\cdot\frac{1}{r}\boldsymbol{p}_r + \boldsymbol{p}_r\cdot\boldsymbol{r}\frac{1}{r^3}\boldsymbol{r}\cdot\boldsymbol{p}_r\right) \\
&\quad - \frac{2\gamma+1}{2c^2}\frac{m_1 - m_2}{m_1 m_2}\boldsymbol{p}_r\cdot(\boldsymbol{r}\cdot\boldsymbol{\nabla}\phi(\boldsymbol{R}))\boldsymbol{p}_r - \frac{\gamma+1}{c^2}\frac{e^2}{8\pi\varepsilon_0 r}\frac{m_2 - m_1}{M}\boldsymbol{r}\cdot\boldsymbol{\nabla}\phi(\boldsymbol{R}) \\
H_{\mathrm{AL,final}} &= \left(1 + (\gamma+1)\frac{\phi(\boldsymbol{R})}{c^2}\right)\frac{\tilde{\boldsymbol{\Pi}}^{\perp}(\boldsymbol{R})}{\varepsilon_0}\cdot\boldsymbol{d} + \frac{1}{2M}\{\boldsymbol{P}\cdot[\boldsymbol{d}\times\boldsymbol{B}(\boldsymbol{R})] + \mathrm{H.c.}\} \\
&\quad - \frac{m_1 - m_2}{4m_1m_2}\{\boldsymbol{p}_r\cdot[\boldsymbol{d}\times\boldsymbol{B}(\boldsymbol{R})] + \mathrm{H.c.}\} \\
&\quad + \frac{1}{8\mu}(\boldsymbol{d}\times\boldsymbol{B}(\boldsymbol{R}))^2 + \frac{1}{2\varepsilon_0}\int \mathrm{d}^3\boldsymbol{x}\left(1 + (\gamma+1)\frac{\phi}{c^2}\right)\boldsymbol{\mathcal{P}}_d^{\perp\,2}(\boldsymbol{x},t) \\
&\quad - \int \mathrm{d}^3\boldsymbol{x}\,(\gamma+1)\,\phi_{\mathrm{el.}}^{(0)}\frac{\boldsymbol{\nabla}\phi}{c^2}\cdot(\tilde{\boldsymbol{\Pi}}^{\perp} + \boldsymbol{\mathcal{P}}_d^{\perp}) \\
H_{\mathrm{L,final}} &= \frac{\varepsilon_0}{2}\int \mathrm{d}^3\boldsymbol{x}\left(1 + (\gamma+1)\frac{\phi}{c^2}\right)\left[(\tilde{\boldsymbol{\Pi}}^{\perp}/\varepsilon_0)^2 + c^2(\boldsymbol{\nabla}\times\boldsymbol{A}^{\perp})^2\right] \\
H_{\mathrm{X}} &= -\frac{(\boldsymbol{P}\cdot\boldsymbol{p}_r)^2}{2M^2\mu c^2} + \frac{e^2}{4\pi\varepsilon_0 r}\frac{(\boldsymbol{P}\cdot\boldsymbol{r}/r)^2}{2M^2c^2} \\
&\quad + \frac{m_1 - m_2}{2\mu M^2c^2}\left\{(\boldsymbol{P}\cdot\boldsymbol{p}_r)\boldsymbol{p}_r^2/\mu - \frac{e^2}{8\pi\varepsilon_0}\left[\frac{1}{r}\boldsymbol{P}\cdot\boldsymbol{p}_r + \frac{1}{r^3}(\boldsymbol{P}\cdot\boldsymbol{r})(\boldsymbol{r}\cdot\boldsymbol{p}_r) + \mathrm{H.c.}\right]\right\} \\
H_{\mathrm{deriv,new}} &= \frac{2\gamma+1}{2Mc^2}[\boldsymbol{P}\cdot(\boldsymbol{r}\cdot\boldsymbol{\nabla}\phi(\boldsymbol{R}))\boldsymbol{p}_r + \mathrm{H.c.}]
\end{aligned}
$$

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The full Pauli Hamiltonian

$$
\begin{aligned}
H_{\text{Pauli}} = & \Big\{ -\frac{1}{2m} - \frac{1}{2mc^2}(\boldsymbol{a}\cdot\boldsymbol{x}) - \frac{1}{4m} R_{0l0m} x^l x^m \Big\} (\boldsymbol{\sigma}\cdot\boldsymbol{D})^2 \\
& + \Big\{ -\frac{1}{4mc^2} a^j - \frac{\mathrm{i}}{4mc^2} a_i \varepsilon^{ij}{}_k \sigma^k + \frac{1}{12m} R_{0l0}{}^j x^l \\
& \quad - \frac{\mathrm{i}}{4m} \varepsilon^{ij}{}_k \sigma^k R_{0l0i} x^l + \frac{1}{3m} R_l{}^j x^l - \frac{2\mathrm{i}c}{3} R^j{}_{l0m} x^l x^m \\
& \quad - \frac{\mathrm{i}}{12m} \varepsilon^{iq}{}_k \sigma^k x^l (R_{ql}{}^j{}_i + R_{qi}{}^j{}_l) + \mathrm{i}(\boldsymbol{\omega}\times\boldsymbol{x})^j \Big\} D_j \\
& - \frac{1}{6m} R^i{}_l{}^j{}_m x^l x^m D_j D_i - qcA_0 - \frac{1}{2}\sigma_p \omega^p + \frac{mc^2}{2} R_{0l0m} x^l x^m \\
& + \frac{1}{8m} R + \frac{1}{4m} R_{00} + m(\boldsymbol{a}\cdot\boldsymbol{x}) + \frac{\mathrm{i}c}{3} R_{0l} x^l \\
& + c\varepsilon^{jb}{}_k \sigma^k x^l \left(\frac{1}{3} R_{0jbl} + \frac{1}{12} R_{bj0l} \right) - \frac{q}{4m^2c^2} \sigma^b \sigma^j D_b E_j \\
& - \frac{1}{8m^3c^2} (\boldsymbol{\sigma}\cdot\boldsymbol{D})^4 + \frac{\mathrm{i}}{4m^2c^2} (\boldsymbol{\omega}\times\boldsymbol{x})^i (\boldsymbol{\sigma}\cdot\boldsymbol{D})^2 D_i \\
& - \frac{\mathrm{i}}{4m^2c^2} \sigma^i \sigma^j (\boldsymbol{\omega}\times\boldsymbol{x})^k D_i D_j D_k \\
& - \frac{\mathrm{i}}{4} x^l (\boldsymbol{\omega}\times\boldsymbol{x})^r (R_{rl} + R_{0r0l}) + \mathrm{O}(c^{-3}) + \mathrm{O}(x^3)
\end{aligned}
$$

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