Time in Newtonian physics from a spacetime perspective

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Introduction

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- This talk: review of Newtonian concept of time from a *spacetime* point of view, clarifying its relation to time in special and general relativity
- Origin in *Newton–Cartan gravity*: geometric reformulation of Newtonian gravity, developed for clearer understanding of relationship to GR (Cartan 1923, Friedrichs 1926/28, Trautman 1960s, Künzle 1970s, Ehlers 1981)

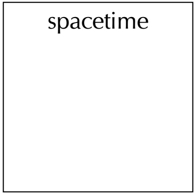
Outline

Newton–Galilei spacetime

Geometry

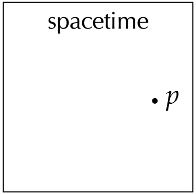
Newtonian physics: infinitesimal temporal structure

In spacetime



Newtonian physics: infinitesimal temporal structure

In spacetime, at an event p



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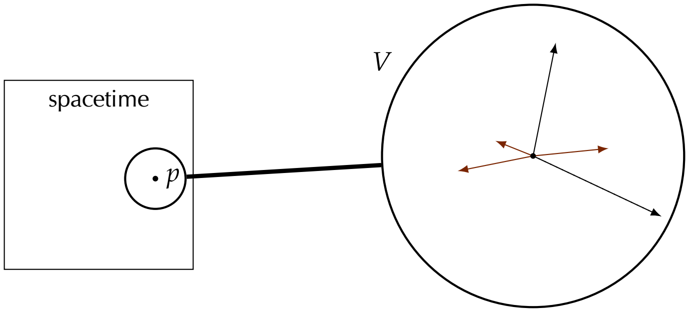

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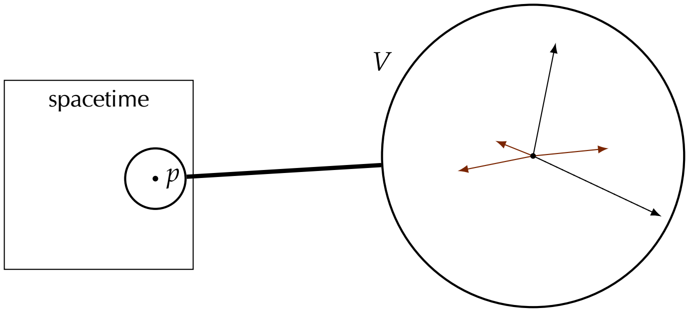

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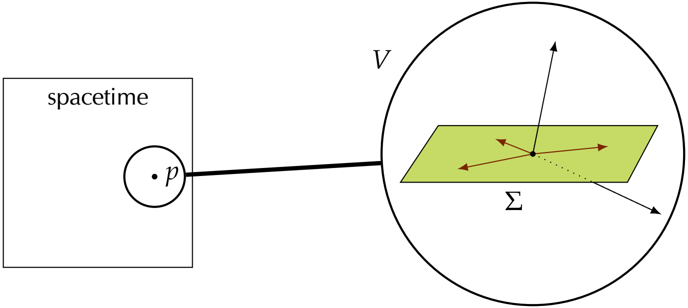

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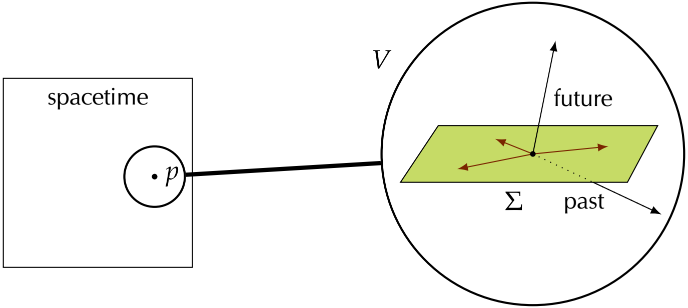

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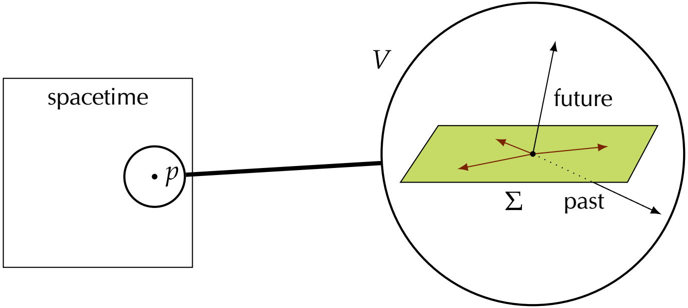

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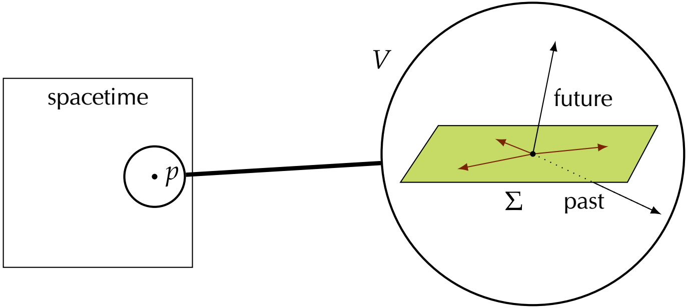

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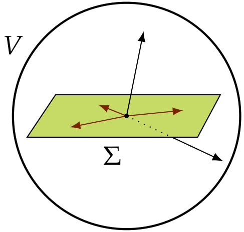
- Described by *clock form* $\tau \in V^*$, $\Sigma = \ker \tau$

Spatial measurements



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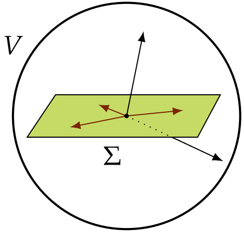

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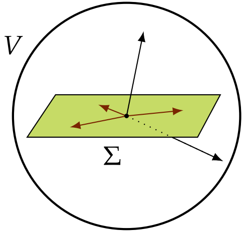

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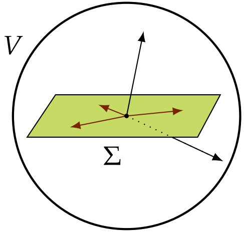

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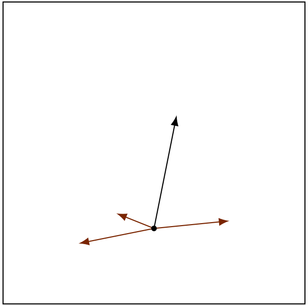
- Space metric $h^{\mu\nu}$, signature $(0{+}{+}{+})$, $h^{\mu\nu}\tau_\nu = 0$

Newton–Galilei spacetime: space and time of Newton



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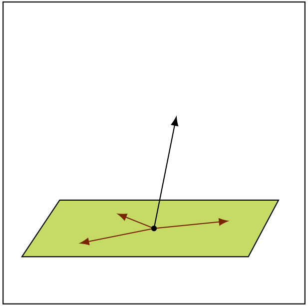

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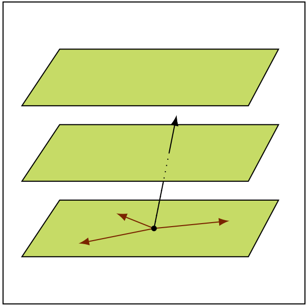

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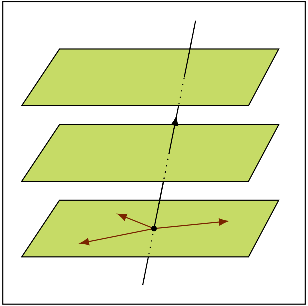

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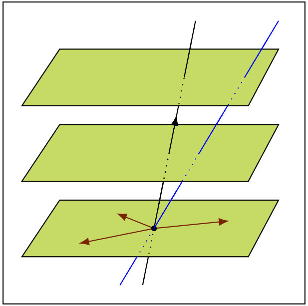

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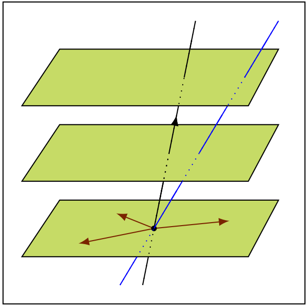

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- Time being absolute means *time durations* and *simultaneity* being absolute.

Galilei transformations

- Galilei transformations are automorphisms of Newton–Galilei spacetime

$$\begin{aligned}\mathrm{IGal} &:= \mathrm{Aut}(A, \tau, h)\\ &= \{\text{affine maps } A \to A \text{ leaving temporal and spatial lengths invariant}\}\\ &= \{(X, v) \in \mathrm{Aff}(A) \cong \mathrm{GL}(V) \ltimes V : X^\top \tau = \tau, (X \otimes X)(h) = h\} \qquad (1)\end{aligned}$$

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- Choosing origin $o \in A$ and reference four-velocity $v \in V$, $\tau(v) = 1$: homogeneous group consists of rotations and boosts

$$\mathsf{Gal} \cong \mathrm{O}(\Sigma) \ltimes \Sigma, \tag{2}$$

acting on spacetime points according to

$$(R,\boldsymbol{k}) \cdot p = o + R(P_v(p-o)) + \tau(p-o)\boldsymbol{k}, \tag{3}$$

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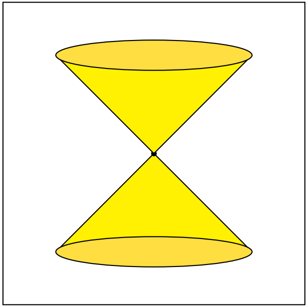
- Pure boost: $\begin{pmatrix} t \\ \boldsymbol{x} \end{pmatrix} \mapsto \begin{pmatrix} t \\ \boldsymbol{x} + t\boldsymbol{k} \end{pmatrix}$, time axis $\left\{ \begin{pmatrix} t \\ \boldsymbol{0} \end{pmatrix} : t \in \mathbb{R} \right\} \mapsto \left\{ \begin{pmatrix} t \\ t\boldsymbol{k} \end{pmatrix} : t \in \mathbb{R} \right\}$ is tilted

The relation to special relativity

Newton–Galilei from Minkowski: causal structure



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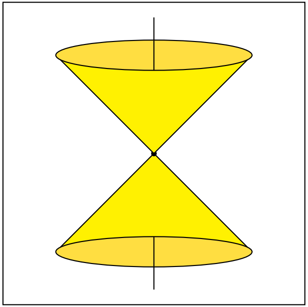

- Minkowski spacetime, reference inertial state of motion

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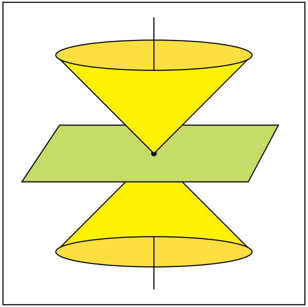

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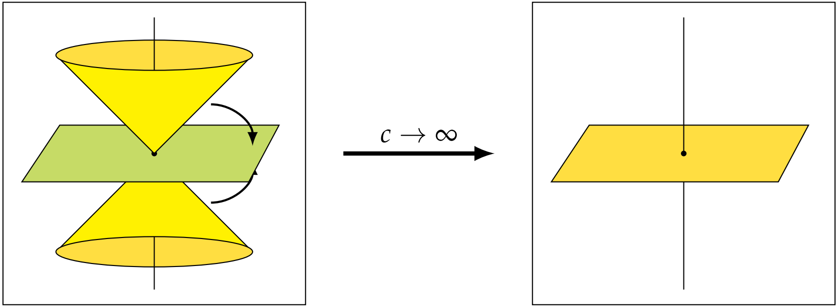

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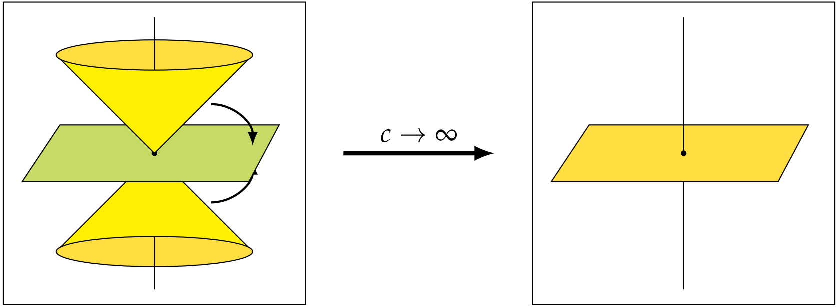

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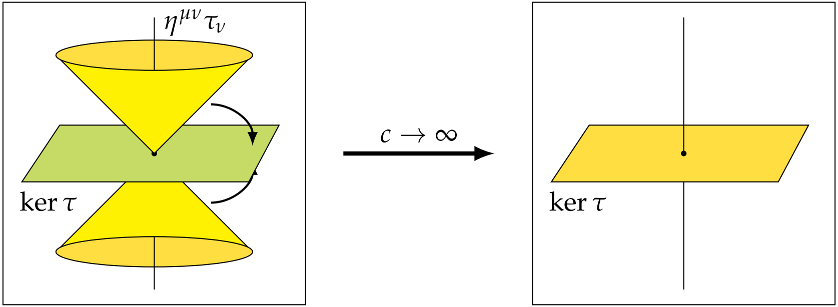

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- Reference: choice of τ! Expand $\eta_{\mu\nu} = -c^2\tau_\mu\tau_\nu + O(c^0)$, $\eta^{\mu\nu} = h^{\mu\nu} + O(c^{-2})$

Newton–Galilei from Minkowski: geometric measurements

Choose τ, assume power series expansion $\eta_{\mu\nu} = -c^2 \tau_\mu \tau_\nu + \mathrm{O}(c^0)$, $\eta^{\mu\nu} = h^{\mu\nu} + \mathrm{O}(c^{-2})$

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- Minkowski proper time of a future-directed timelike vector v:

$$\begin{aligned} c^{-1}\sqrt{-\eta(v,v)} &= \sqrt{-c^{-2}\eta(v,v)} = \sqrt{(\tau(v))^2 + \mathrm{O}(c^{-2})} \\ &= \tau(v) + \mathrm{O}(c^{-2}) \xrightarrow{c\to\infty} \tau(v) \end{aligned} \tag{4}$$

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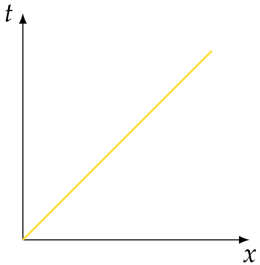
- Same for spatial metric: $\eta(v,w) \xrightarrow{c\to\infty} {}^{(\Sigma)}h(v,w)$ for v, w spacelike

Galilei from Poincaré: boosts



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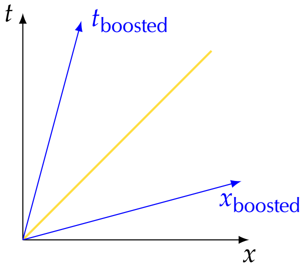

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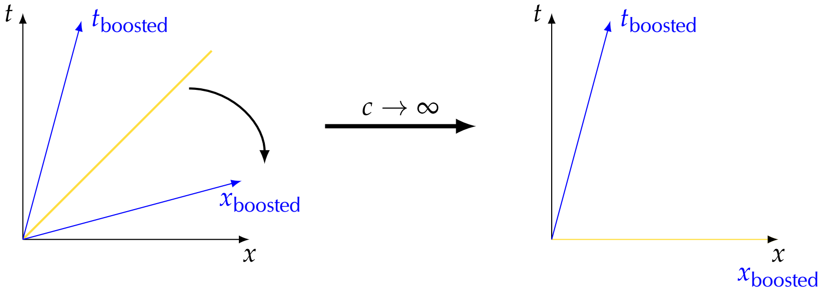

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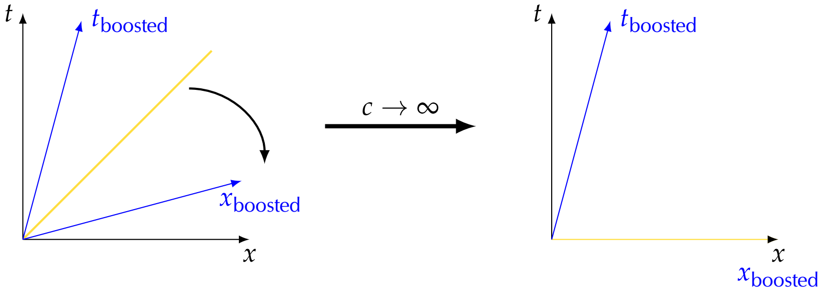

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- Lorentz boost $\begin{pmatrix} t \\ x \end{pmatrix} \mapsto \frac{1}{\sqrt{1-k^2/c^2}} \begin{pmatrix} 1 & \frac{k}{c^2} \\ k & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \xrightarrow{c\to\infty} \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$ Galilei boost

Galilei from Poincaré: group contraction

İnönü–Wigner contraction from Poincaré to Galilei group:

- Poincaré algebra: Translation generators P^μ, Lorentz transformation generators $J_{\mu\nu}$, brackets

$$[P_\mu, P_\nu] = 0, \tag{5a}$$

$$[J_{\mu\nu}, P_\rho] = \eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu \ , \tag{5b}$$

$$[J_{\mu\nu}, J_{\rho\sigma}] = \big(\eta_{\mu\rho} J_{\nu\sigma} - (\mu \leftrightarrow \nu)\big) - (\rho \leftrightarrow \sigma) \tag{5c}$$

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- Time translation generator $cP^0 =: -H$, boost generator $c^{-1} J_{a0} =: B_a$
- Non-trivial Poincaré relations are

$$[B_a, H] = P_a \,, [B_a, P_b] = c^{-2} \delta_{ab} H, [B_a, B_b] = -c^{-2} J_{ab} \,, \tag{6a}$$
$$[J_{ab}, P_c] = \delta_{ac} P_b - \delta_{bc} P_a \,, [J_{ab}, B_c] = \delta_{ac} B_b - \delta_{bc} B_a \,, \tag{6b}$$
$$[J_{ab}, J_{cd}] = \big(\eta_{ac} J_{bd} - (a \leftrightarrow b)\big) - (c \leftrightarrow d) \tag{6c}$$

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- Non-trivial Poincaré relations are

$$[B_a, H] = P_a \ , [B_a, P_b] = \cancelto{0}{c^{-2} \delta_{ab} H} , [B_a, B_b] = \cancelto{0}{-c^{-2} J_{ab}} \ , \tag{6a}$$
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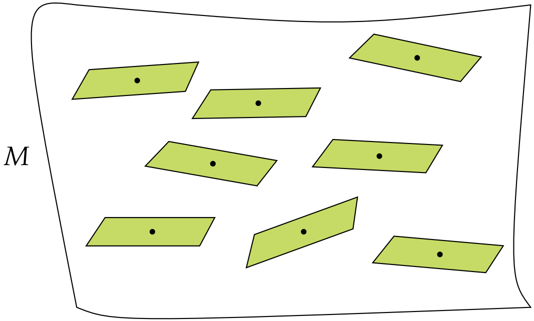
$$[J_{ab}, J_{cd}] = \big(\eta_{ac} J_{bd} - (a \leftrightarrow b)\big) - (c \leftrightarrow d) \tag{6c}$$

- Limit $c \to \infty$ gives inhomogeneous Galilei algebra!

General Newtonian spacetime

Geometry

Galilei manifolds



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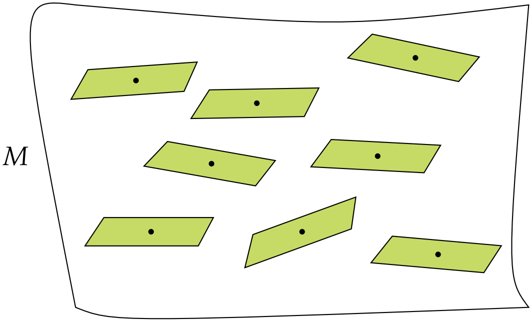

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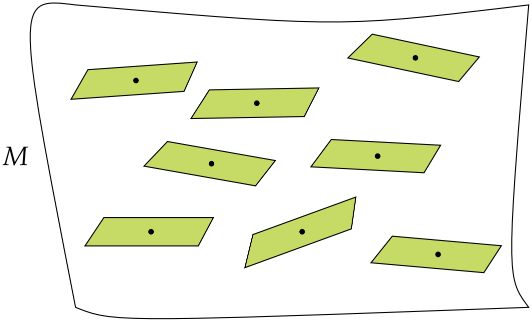

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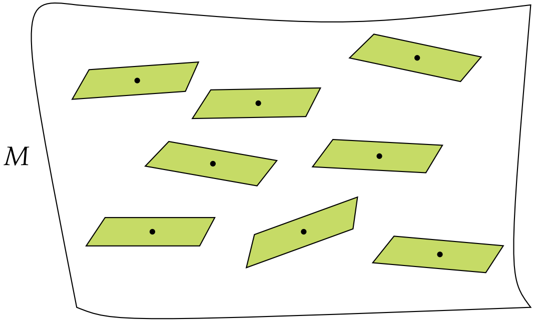

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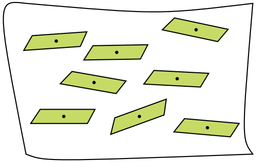
- (hom.) Galilei group as structure group for reduction of the frame bundle—*local Galilei invariance*

Absolute time?



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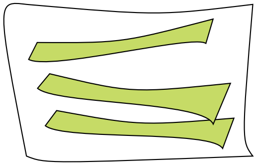

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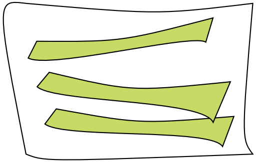

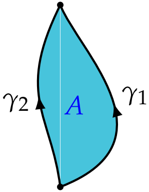

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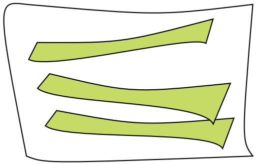

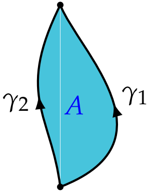

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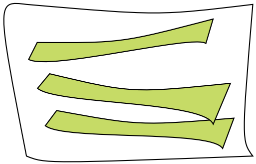

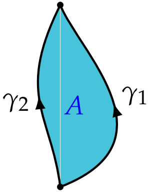

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- $d\tau = 0 \implies$ locally $\tau = dt$, with time function t

Galilei connections

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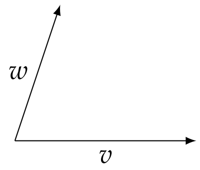
- Newton–Cartan gravity: describe Newtonian gravity by curved ∇ (i.e. parallel transport has holonomy)

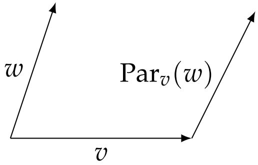
Torsion and time



- Torsion T of connection ∇: measures how infinitesimal parallelograms fail to close

Curvature and space

Torsion and time



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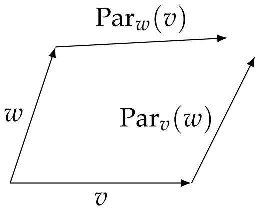

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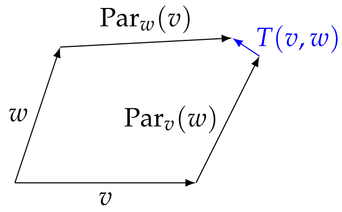

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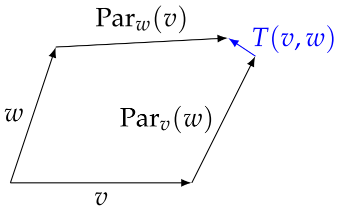

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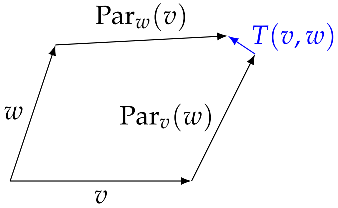

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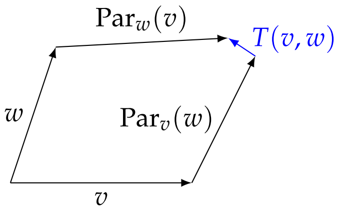

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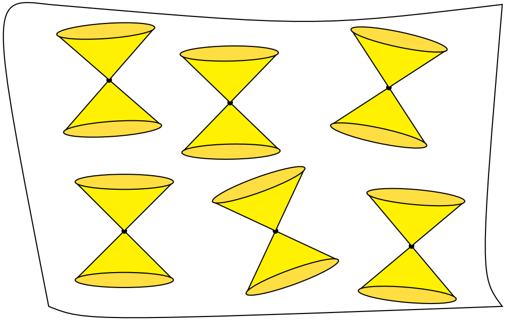
Curvature and space

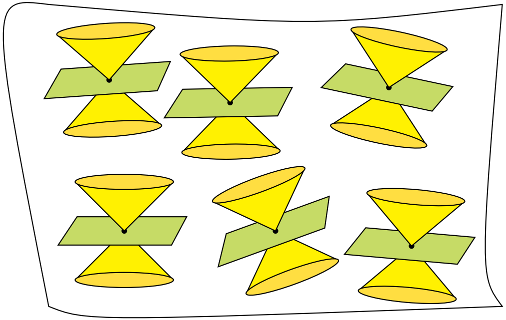
The relation to general relativity

Newtonian limits of Lorentzian manifolds



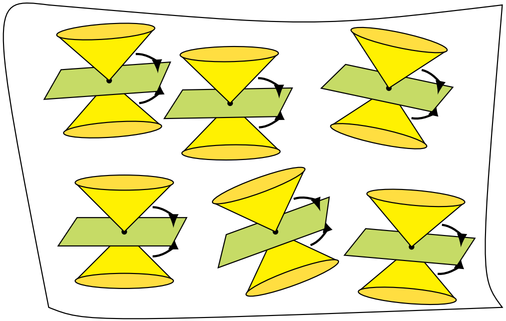
- Lorentzian manifold

Newtonian limits of Lorentzian manifolds



- Lorentzian manifold, reference state of motion

Newtonian limits of Lorentzian manifolds



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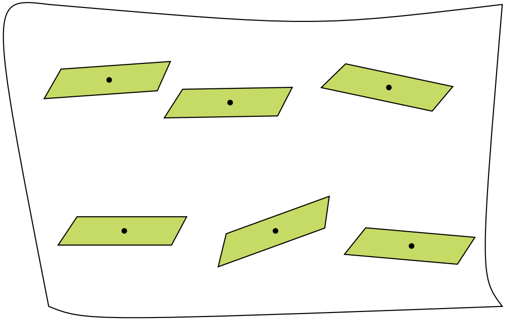

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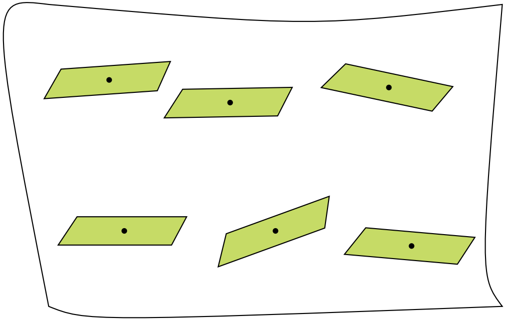

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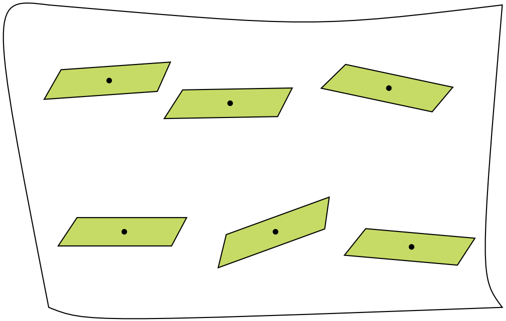

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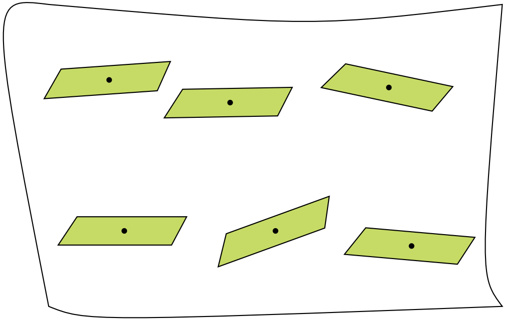

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- Lorentzian proper time:

$$c^{-1} \int_\gamma \sqrt{-g(\cdot,\cdot)} = \int_\gamma \tau + \mathrm{O}(c^{-2}) \tag{8}$$

The Newtonian limit and absolute time

Power series expansion of Lorentzian metric: $g_{\mu\nu} = -c^2\tau_\mu\tau_\nu + \mathrm{O}(c^0)$, $g^{\mu\nu} = h^{\mu\nu} + \mathrm{O}(c^{-2})$

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- Trace-reversed Einstein equation for (M, g) goes over to Newton–Cartan field equation for (M, τ, h, ∇)

Example: the Newtonian limit of Schwarzschild

$$g = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 \mathrm{d}t^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2) \tag{10a}$$

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- ∇ gives Newtonian trajectories
- Same limit for Kerr (assuming regularity outside horizon for all values of c)

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Example: another 'Newtonian' limit of Schwarzschild

Consider Schwarzschild radius r_S as c-independent $\rightsquigarrow$ near-horizon / strong-gravity limit

$$g = -\left(1 - \frac{r_S}{r}\right) c^2 \mathrm{d}t^2 + \left(1 - \frac{r_S}{r}\right)^{-1} \mathrm{d}r^2 + r^2 (\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2) \tag{12a}$$

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- $\tau = \sqrt{1-\frac{r_S}{r}}\mathrm{d}t$ captures gravitational time dilation
- TNC trajectories *after taking* $c \to \infty$ yield perihelion precession!

Conclusion

- Newtonian spacetime structure is captured by a degenerate metric structure (τ, h) and a compatible connection.
- Having an absolute notion of time is stronger than having an absolute notion of simultaneity, and neither is necessary for (pointwise) Newtonian causality.
- Standard Newtonian physics, i.e. locally Galilei-relativistic physics with absolute time, is only a *special* limiting case of locally Poincaré-relativistic physics.
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Many thanks for your attention!

Appendix: Details

Curvature and space

- Newton–Cartan gravity: Galilei manifold (M, τ, h) with absolute time ($\mathrm{d}\tau = 0$) and torsion-free Galilei connection ∇ (plus $R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma} = R^{\nu}{}_{\sigma}{}^{\mu}{}_{\rho}$)
- Newton–Cartan field equation:

$$R_{\mu\nu} = 4\pi G \rho \tau_\mu \tau_\nu \tag{13}$$

- $\implies$ spacetime Ricci tensor vanishes on spacelike vectors
- Compatibility with h, torsion-freeness $\implies$ connection induced by ∇ on space is Levi-Civita connection
- $\therefore$ field equation $\implies$ space is Ricci-flat $\overset{\text{3d}}{\implies}$ space is flat

Back