The classical analogue of the Newton–Wigner theorem

Philip K Schwartz

Institut für Theoretische Physik
Leibniz Universität Hannover

Seminar on Relativistic Localisation, 18th May 2022

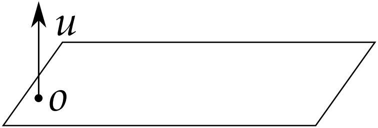
Outline

P K S, D Giulini; *Classical perspectives on the Newton–Wigner position observable*, `arXiv:2004.09723`, Int J Geom Methods Mod Phys **17** 2050176 (2020)

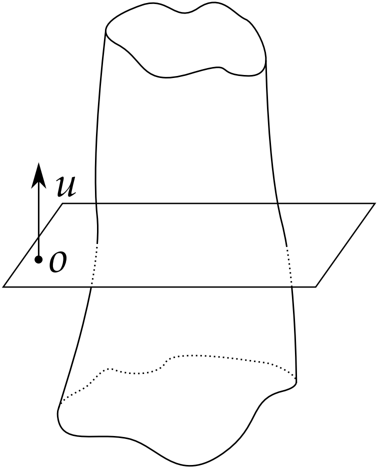
Motivation: The quantum Newton–Wigner theorem

P K S, D Giulini; *Classical perspectives on the Newton–Wigner position observable*, `arXiv:2004.09723`, Int J Geom Methods Mod Phys **17** 2050176 (2020)

Localising a Poincaré invariant quantum system

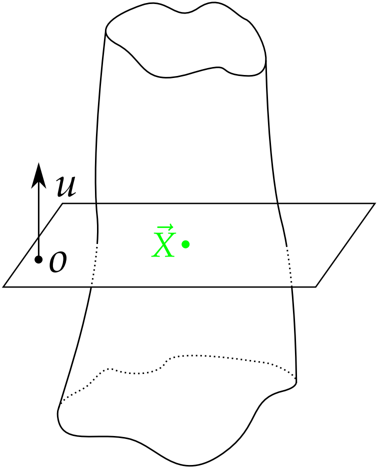


Localising a Poincaré invariant quantum system



Localising a Poincaré invariant quantum system

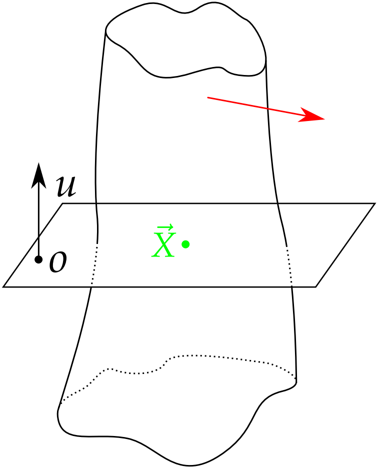
Possible demands on a position observable:



Localising a Poincaré invariant quantum system

Possible demands on a position observable:

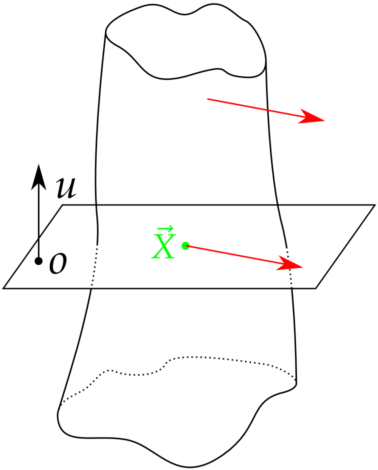
- Translation covariance: Translations of system



Localising a Poincaré invariant quantum system

Possible demands on a position observable:

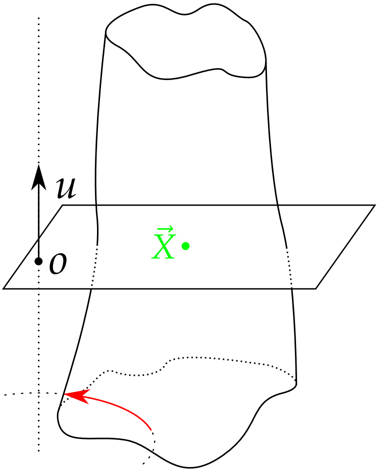
- Translation covariance: Translations of system translate position



Localising a Poincaré invariant quantum system

Possible demands on a position observable:

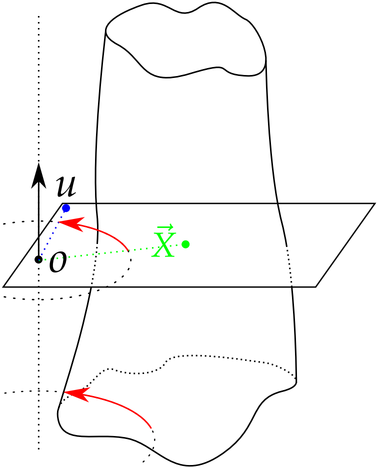
- Translation covariance: Translations of system translate position
- Rotation covariance: Rotations of system



Localising a Poincaré invariant quantum system

Possible demands on a position observable:

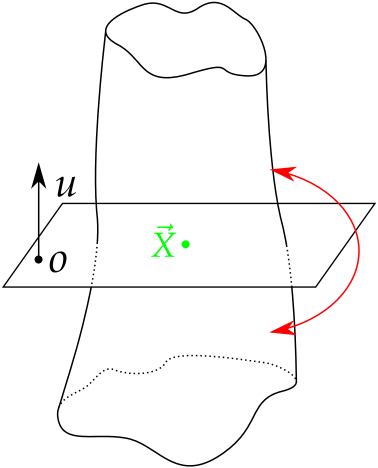
- Translation covariance: Translations of system translate position
- Rotation covariance: Rotations of system rotate position



Localising a Poincaré invariant quantum system

Possible demands on a position observable:

- Translation covariance: Translations of system translate position
- Rotation covariance: Rotations of system rotate position
- Localisation on given hyperplane: Time-reversal invariant



The quantum Newton–Wigner theorem

Theorem (Quantum Newton–Wigner theorem)

For an elementary Poincaré invariant quantum system with timelike four-momentum, there is a unique 3-tuple $\vec{X}$ of self-adjoint operators on the Hilbert space that

1. *satisfies some regularity condition,*
2. *has commuting components,* $[X^a, X^b] = 0$,
3. *satisfies the canonical commutation relations* $[X^a, P_b] = \mathrm{i}\hbar\delta^a_b$ *with the generators of spatial translations with respect to* u,
4. *transforms 'as a (position) vector' under spatial rotations with respect to* u*, i.e. satisfies* $[J_{ab}, X^c] = \mathrm{i}\hbar(\delta^c_a X_b - \delta^c_b X_a)$*, and*
5. *is invariant under time reversal.*

T D Newton, E P Wigner; *Localized states for elementary systems*, Rev Mod Phys **21** 400–406 (1949)

The quantum Newton–Wigner theorem – explicit form

In terms of the Poincaré generators, the Newton–Wigner position has the form

$$X^a = -\frac{1}{2}\left((P^0)^{-1}J_{a0} + J_{a0}(P^0)^{-1}\right) + \frac{\varepsilon_{abc}P^b W^c}{P^0 mc(mc + P^0)}, \quad (1)$$

where $m = \sqrt{-P_\mu P^\mu}/c$ is the mass and $W_\mu = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}P^\nu J^{\rho\sigma}$ is the Pauli–Lubański vector (Casimir operators of Poincaré algebra).

Classical elementary systems

P K S, D Giulini; *Classical perspectives on the Newton–Wigner position observable*, `arXiv:2004.09723`, Int J Geom Methods Mod Phys **17** 2050176 (2020)

Definition and classification

P K S, D Giulini; *Classical perspectives on the Newton–Wigner position observable*, `arXiv:2004.09723`, Int J Geom Methods Mod Phys **17** 2050176 (2020)

Conventions for Minkowski space

- $(-+++)$
- M affine space
- V associated vector space
- Metric $\eta \colon V \times V \to \mathbb{R}$
- $v \cdot w := \eta(v, w)$
- Fix arbitrary origin $o \in M$
- Fix ONB $\{u = e_0, \dots, e_3\}$ of V

Classical Poincaré-invariant systems

Definition

A *classical Poincaré-invariant system* (Γ, ω, Φ) is a phase space (Γ, ω) (i.e. a symplectic manifold) together with a Poisson action Φ of the identity component $\mathcal{P}_+^\uparrow$ of the Poincaré group.

- Symplectic action is sufficient in spacetime dimension > 2

Classical Poincaré-invariant systems

Definition

A *classical Poincaré-invariant system* (Γ, ω, Φ) is a phase space (Γ, ω) (i.e. a symplectic manifold) together with a Poisson action Φ of the identity component $\mathcal{P}_+^\uparrow$ of the Poincaré group.

- Symplectic action is sufficient in spacetime dimension > 2
- Poincaré generators (components of momentum map):

$$\begin{array}{lll} P_\mu \text{ gen. of translations} & \rightarrow & \text{'four-momentum'} \\ J_{ab} \text{ gen. of rotations wrt. } o & \multirow{2}{*}{\}} & \multirow{2}{*}{'angular momentum tensor'} \\ J_{a0} \text{ gen. of boosts wrt. } o & & \end{array}$$

Classical Poincaré-invariant systems

Definition

A *classical Poincaré-invariant system* (Γ, ω, Φ) is a phase space (Γ, ω) (i.e. a symplectic manifold) together with a Poisson action Φ of the identity component $\mathcal{P}_+^\uparrow$ of the Poincaré group.

- Symplectic action is sufficient in spacetime dimension > 2
- Poincaré generators (components of momentum map):
  P_μ gen. of translations $\rightarrow$ 'four-momentum'
  J_{ab} gen. of rotations wrt. o } 'angular momentum tensor'
  J_{a0} gen. of boosts wrt. o }
- Generators as phase space functions: $P \colon \Gamma \to V$, $J \colon \Gamma \to \bigwedge^2 V^*$

Mass and spin

Let (Γ, ω, Φ) be a classical Poincaré-invariant system.

- For P causal: The system's *mass* is $m := \sqrt{-P^2}/c$.

Mass and spin

Let (Γ, ω, Φ) be a classical Poincaré-invariant system.

- For P causal: The system's *mass* is $m := \sqrt{-P^2}/c$.
- The *Pauli–Lubański vector* is $W := (*(P^\flat \wedge J))^\sharp$, i.e. $W_\mu = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}P^\nu J^{\rho\sigma}$.

Mass and spin

Let (Γ, ω, Φ) be a classical Poincaré-invariant system.

- For P causal: The system's *mass* is $m := \sqrt{-P^2}/c$.
- The *Pauli–Lubański vector* is $W := (*(P^\flat \wedge J))^\sharp$, i.e. $W_\mu = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}P^\nu J^{\rho\sigma}$.
- For P timelike: $W/(mc)$ generates rotations in the momentum rest frame, and …

Mass and spin

Let (Γ, ω, Φ) be a classical Poincaré-invariant system.

- For P causal: The system's *mass* is $m := \sqrt{-P^2}/c$.
- The *Pauli–Lubański vector* is $W := (*(P^\flat \wedge J))^\sharp$, i.e. $W_\mu = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}P^\nu J^{\rho\sigma}$.
- For P timelike: $W/(mc)$ generates rotations in the momentum rest frame, and ...
- ... the system's *spin* is $S := \sqrt{W^2}/(mc)$.

Elementary systems

Definition

A *classical elementary system* is a classical Poincaré-invariant system (Γ, ω, Φ) where the action Φ is transitive.

- Equivalent: Γ has no proper invariant subset!
- $\rightsquigarrow$ direct translation of the quantum case (irreducibility)

Classification of elementary systems

- Classical elementary systems classified by Arens (1971)
- Classification in terms of character of P and W
- Phase spaces constructed as coadjoint orbits of $\mathcal{P}_+^\uparrow$
- Relevant for us: P timelike

Richard Arens; *Classical Lorentz Invariant Particles*, J Math Phys **12** 2415–2422 (1971)

Classification of elementary systems, timelike P

Theorem (Phase space of a classical elementary system)

Any classical elementary system with timelike four-momentum is equivalent to precisely one of the following two cases:

Classification of elementary systems, timelike P

Theorem (Phase space of a classical elementary system)

Any classical elementary system with timelike four-momentum is equivalent to precisely one of the following two cases:

1. *(Spin zero, one parameter $m \in \mathbb{R}_+$)*
   - *Phase space* $(\Gamma = T^*\mathbb{R}^3, \omega = \mathrm{d}x^a \wedge \mathrm{d}p_a)$
   - *Poincaré generators:*

$$\text{spatial translations} \quad P_a = p_a \tag{2a}$$

$$\text{time translation} \quad P_0 = -\sqrt{m^2c^2 + \vec{p}^2} \tag{2b}$$

$$\text{rotations} \quad J_{ab} = x_a p_b - x_b p_a \tag{2c}$$

$$\text{boosts} \quad J_{a0} = P_0 x_a \tag{2d}$$

Classification of elementary systems, timelike P

Theorem (Phase space of a classical elementary system)

2. *(Spin non-zero, two parameters $m, S \in \mathbb{R}_+$)*
   - *Phase space* $(\Gamma = T^*\mathbb{R}^3 \times S^2, \omega = \mathrm{d}x^a \wedge \mathrm{d}p_a + S \cdot \mathrm{d}\Omega^2)$

Classification of elementary systems, timelike P

Theorem (Phase space of a classical elementary system)

2. *(Spin non-zero, two parameters $m, S \in \mathbb{R}_+$)*
   - *Phase space* $(\Gamma = T^*\mathbb{R}^3 \times S^2, \omega = \mathrm{d}x^a \wedge \mathrm{d}p_a + S \cdot \mathrm{d}\Omega^2)$
   - $\hat{s} \colon \Gamma \to S^2 \subset \mathbb{R}^3$*, spin vector observable* $\vec{s} := S \cdot \hat{s}$*, satisfying*

$$\{s_a, s_b\} = {}^{(3)}\varepsilon_{abc} s^c \tag{3}$$

Classification of elementary systems, timelike P

Theorem (Phase space of a classical elementary system)

2. *(Spin non-zero, two parameters $m, S \in \mathbb{R}_+$)*
   - *Phase space* $(\Gamma = T^*\mathbb{R}^3 \times S^2, \omega = \mathrm{d}x^a \wedge \mathrm{d}p_a + S \cdot \mathrm{d}\Omega^2)$
   - $\hat{s} \colon \Gamma \to S^2 \subset \mathbb{R}^3$*, spin vector observable* $\vec{s} := S \cdot \hat{s}$*, satisfying*

$$\{s_a, s_b\} = {}^{(3)}\varepsilon_{abc} s^c \tag{3}$$

   - *Poincaré generators:*

$$\text{spatial translations} \quad P_a = p_a \tag{4a}$$

$$\text{time translation} \quad P_0 = -\sqrt{m^2c^2 + \vec{p}^2} \tag{4b}$$

$$\text{rotations} \quad J_{ab} = x_a p_b - x_b p_a + {}^{(3)}\varepsilon_{abc} s^c \tag{4c}$$

$$\text{boosts} \quad J_{a0} = P_0 x_a - \frac{(\vec{p} \times \vec{s})_a}{mc - P_0} \tag{4d}$$

Structure

P K S, D Giulini; *Classical perspectives on the Newton–Wigner position observable*, arXiv:2004.09723, Int J Geom Methods Mod Phys **17** 2050176 (2020)

A complete involutive set

Define $\Gamma^* := \Gamma \setminus \{|\vec{P}| = 0\}$ and $\hat{P} := \frac{\vec{P}}{|\vec{P}|} : \Gamma^* \to S^2$.

▸ Complete involutive sets

A complete involutive set

Define $\Gamma^* := \Gamma \setminus \{|\vec{P}| = 0\}$ and $\hat{P} := \frac{\vec{P}}{|\vec{P}|} : \Gamma^* \to S^2$.

Lemma

For a classical elementary system with timelike four-momentum, the functions $P_a, \hat{P} \cdot \vec{s}$ (or just the P_a in the case of zero spin) form a complete involutive set on Γ^* (or the whole of Γ in the case of zero spin).

▸ Complete involutive sets

Momentum and spin are vectors

Lemma

For a classical elementary system with timelike four-momentum, $\vec{P}$ and $\vec{s}$ are invariant under translations and 'transform as vectors' under spatial rotations, i.e. for $\vec{V} = \vec{P}, \vec{s}$, we have

$$\{P_a, V_b\} = 0, \quad \{J_{ab}, V_c\} = \delta_{ac} V_b - \delta_{bc} V_a \ . \tag{5}$$

Invariant functions

Lemma (Invariant functions)

Consider some open subset $\tilde{\Gamma}$ *of* $\Gamma^* = \Gamma \setminus \{|\vec{P}| = 0\}$. *Let* f *be an* $\mathbb{R}$*-valued* C^1 *function defined on* $\tilde{\Gamma}$ *that is invariant under spatial translations and rotations, i.e.* $\{P_a, f\} = 0 = \{J_{ab}, f\}$. *Then* f *is a function of* $|\vec{P}|, \hat{P} \cdot \vec{s}$.

Invariant functions

Lemma (Invariant functions)

Consider some open subset $\tilde{\Gamma}$ of $\Gamma^* = \Gamma \setminus \{|\vec{P}| = 0\}$. Let f be an $\mathbb{R}$-valued C^1 function defined on $\tilde{\Gamma}$ that is invariant under spatial translations and rotations, i.e. $\{P_a, f\} = 0 = \{J_{ab}, f\}$. Then f is a function of $|\vec{P}|, \hat{P} \cdot \vec{s}$.

Proof.

f Poisson-commutes with $\vec{P}$ and J_{ab}. Therefore it also Poisson-commutes with $\vec{P}$ and $\frac{1}{2}{}^{(3)}\varepsilon^{abc}\hat{P}_a J_{bc} = \hat{P} \cdot \vec{s}$.

Invariant functions

Lemma (Invariant functions)

Consider some open subset $\tilde{\Gamma}$ *of* $\Gamma^* = \Gamma \setminus \{|\vec{P}| = 0\}$. *Let* f *be an* $\mathbb{R}$*-valued* C^1 *function defined on* $\tilde{\Gamma}$ *that is invariant under spatial translations and rotations, i.e.* $\{P_a, f\} = 0 = \{J_{ab}, f\}$. *Then* f *is a function of* $|\vec{P}|, \hat{P} \cdot \vec{s}$.

Proof.

f Poisson-commutes with $\vec{P}$ and J_{ab}. Therefore it also Poisson-commutes with $\vec{P}$ and $\frac{1}{2}{}^{(3)}\varepsilon^{abc}\hat{P}_a J_{bc} = \hat{P} \cdot \vec{s}$. Now $\vec{P}, \hat{P} \cdot \vec{s}$ form a complete involutive set on Γ^*, and thus f is a function of $\vec{P}, \hat{P} \cdot \vec{s}$.

Invariant functions

Lemma (Invariant functions)

Consider some open subset $\tilde{\Gamma}$ of $\Gamma^* = \Gamma \setminus \{|\vec{P}| = 0\}$. Let f be an $\mathbb{R}$-valued C^1 function defined on $\tilde{\Gamma}$ that is invariant under spatial translations and rotations, i.e. $\{P_a, f\} = 0 = \{J_{ab}, f\}$. Then f is a function of $|\vec{P}|, \hat{P} \cdot \vec{s}$.

Proof.

f Poisson-commutes with $\vec{P}$ and J_{ab}. Therefore it also Poisson-commutes with $\vec{P}$ and $\frac{1}{2}{}^{(3)}\varepsilon^{abc}\hat{P}_a J_{bc} = \hat{P} \cdot \vec{s}$. Now $\vec{P}, \hat{P} \cdot \vec{s}$ form a complete involutive set on Γ^*, and thus f is a function of $\vec{P}, \hat{P} \cdot \vec{s}$. Since f and $\hat{P} \cdot \vec{s}$ are rotation invariant, f must be a function of $|\vec{P}|, \hat{P} \cdot \vec{s}$.

Invariant functions

Lemma (Invariant functions)

Consider some open subset $\tilde{\Gamma}$ of $\Gamma^* = \Gamma \setminus \{|\vec{P}| = 0\}$. Let f be an $\mathbb{R}$-valued C^1 function defined on $\tilde{\Gamma}$ that is invariant under spatial translations and rotations, i.e. $\{P_a, f\} = 0 = \{J_{ab}, f\}$. Then f is a function of $|\vec{P}|, \hat{P} \cdot \vec{s}$.

Proof.

f Poisson-commutes with $\vec{P}$ and J_{ab}. Therefore it also Poisson-commutes with $\vec{P}$ and $\frac{1}{2}{}^{(3)}\varepsilon^{abc}\hat{P}_a J_{bc} = \hat{P} \cdot \vec{s}$. Now $\vec{P}, \hat{P} \cdot \vec{s}$ form a complete involutive set on Γ^*, and thus f is a function of $\vec{P}, \hat{P} \cdot \vec{s}$. Since f and $\hat{P} \cdot \vec{s}$ are rotation invariant, f must be a function of $|\vec{P}|, \hat{P} \cdot \vec{s}$. □

Time reversal

- Need to implement *time reversal* T_u wrt. hyperplane through o and $\perp$ to $u = e_0$, i.e. extend action to $\mathcal{P}_+^\uparrow \cup \mathcal{P}_-^{\downarrow(u)}$

Time reversal

- Need to implement *time reversal* T_u wrt. hyperplane through o and $\perp$ to $u = e_0$, i.e. extend action to $\mathcal{P}_+^\uparrow \cup \mathcal{P}_-^{\downarrow(u)}$
- From general considerations:

$$P_a \circ T_u = -P_a \,, J_{ab} \circ T_u = -J_{ab} \,, J_{a0} \circ T_u = J_{a0} \,, P_0 \circ T_u = P_0 \tag{6}$$

Time reversal

- Need to implement *time reversal* T_u wrt. hyperplane through o and $\perp$ to $u = e_0$, i.e. extend action to $\mathcal{P}_+^\uparrow \cup \mathcal{P}_-^{\downarrow(u)}$
- From general considerations:

$$P_a \circ T_u = -P_a \ , J_{ab} \circ T_u = -J_{ab} \ , J_{a0} \circ T_u = J_{a0} \ , P_0 \circ T_u = P_0 \tag{6}$$

Lemma

For an elementary system with timelike four-momentum, time reversal as above is given by

$$T_u \colon (\vec{x}, \vec{p}, \hat{s}) \mapsto (\vec{x}, -\vec{p}, -\hat{s}). \tag{7}$$

The classical Newton–Wigner theorem

P K S, D Giulini; *Classical perspectives on the Newton–Wigner position observable*, `arXiv:2004.09723`, Int J Geom Methods Mod Phys **17** 2050176 (2020)

Statement

Theorem (Classical Newton–Wigner theorem)

For a classical elementary system with timelike four-momentum, there is a unique $\mathbb{R}^3$-valued phase space function $\vec{X}$ that

1. *is C^1,*
2. *has Poisson-commuting components,*
3. *satisfies the canonical Poisson relations $\{X^a, P_b\} = \delta^a_b$ with the generators of spatial translations with respect to $u = e_0$,*
4. *transforms 'as a (position) vector' under spatial rotations with respect to $u = e_0$, i.e. satisfies $\{J_{ab}, X^c\} = \delta^c_a X_b - \delta^c_b X_a$, and*
5. *is invariant under time reversal wrt. the hyperplane through o and $\perp$ to $u = e_0$, i.e. satisfies $\vec{X} \circ T_u = \vec{X}$.*

As in the quantum case, $\vec{X}$ is given by (1).

Proof: setting up

- We fix a classical elementary system with timelike four-momentum, with phase space Γ.
- Reminder: $\Gamma = T^*\mathbb{R}^3[\times S^2] \ni (\vec{x}, \vec{p}[, \hat{s}])$
- $\vec{x}$ satisfies all conditions on $\vec{X}$
- We need to show uniqueness
- Proof follows the quantum-mechanical proof by Jordan (1980)

Thomas F Jordan; *Simple derivation of the Newton–Wigner position operator*, J Math Phys **21** 2028–2032 (1980)

Proof: step 1

- Let $\vec{X}$ be as in the theorem, and set $\vec{d} := \vec{X} - \vec{x}$.

Proof: step 1

- Let $\vec{X}$ be as in the theorem, and set $\vec{d} := \vec{X} - \vec{x}$.
- $\vec{d}$ ...

Proof: step 1

- Let $\vec{X}$ be as in the theorem, and set $\vec{d} := \vec{X} - \vec{x}$.
- $\vec{d}$ ...
  - is C^1,

Proof: step 1

- Let $\vec{X}$ be as in the theorem, and set $\vec{d} := \vec{X} - \vec{x}$.
- $\vec{d}$ ...
  - is C^1,
  - is invariant under translations (i.e. $\{d^a, P_b\} = 0$),

Proof: step 1

- Let $\vec{X}$ be as in the theorem, and set $\vec{d} := \vec{X} - \vec{x}$.
- $\vec{d}$ ...
  - is C^1,
  - is invariant under translations (i.e. $\{d^a, P_b\} = 0$),
  - transforms as a vector under spatial rotations (i.e. $\{J_{ab}, d^c\} = \delta^c_a d_b - \delta^c_b d_a$),

Proof: step 1

- Let $\vec{X}$ be as in the theorem, and set $\vec{d} := \vec{X} - \vec{x}$.
- $\vec{d}$ ...
  - is C^1,
  - is invariant under translations (i.e. $\{d^a, P_b\} = 0$),
  - transforms as a vector under spatial rotations (i.e. $\{J_{ab}, d^c\} = \delta^c_a d_b - \delta^c_b d_a$),
  - and is invariant under time reversal (i.e. $\vec{d} \circ T_u = \vec{d}$).

Proof: step 1

- Let $\vec{X}$ be as in the theorem, and set $\vec{d} := \vec{X} - \vec{x}$.
- $\vec{d}$ ...
  - is C^1,
  - is invariant under translations (i.e. $\{d^a, P_b\} = 0$),
  - transforms as a vector under spatial rotations (i.e. $\{J_{ab}, d^c\} = \delta_a^c d_b - \delta_b^c d_a$),
  - and is invariant under time reversal (i.e. $\vec{d} \circ T_u = \vec{d}$).

Lemma

These properties imply $\vec{d} \cdot \vec{P} = 0$.

Proof: step 1

Lemma

If $\vec{A}\colon \Gamma \to \mathbb{R}^3$ is C^1, invariant under translations, a vector under spatial rotations and invariant under time reversal, then $\vec{A} \cdot \vec{P} = 0$.

Proof: step 1

Lemma

If $\vec{A}\colon \Gamma \to \mathbb{R}^3$ is C^1, invariant under translations, a vector under spatial rotations and invariant under time reversal, then $\vec{A} \cdot \vec{P} = 0$.

Proof.

$\vec{A} \cdot \vec{P}$ is invariant under translations and rotations.

Proof: step 1

Lemma

If $\vec{A}\colon \Gamma \to \mathbb{R}^3$ is C^1, invariant under translations, a vector under spatial rotations and invariant under time reversal, then $\vec{A} \cdot \vec{P} = 0$.

Proof.

$\vec{A} \cdot \vec{P}$ is invariant under translations and rotations. Therefore, $\vec{A} \cdot \vec{P}\big|_{\Gamma^*}$ is a function of $|\vec{P}|, \hat{P} \cdot \vec{s}$, i.e.

$$\vec{A} \cdot \vec{P}\Big|_{\Gamma^*} = F(|\vec{P}|, \hat{P} \cdot \vec{s}). \tag{8}$$

Proof: step 1

Lemma

If $\vec{A}\colon \Gamma \to \mathbb{R}^3$ is C^1, invariant under translations, a vector under spatial rotations and invariant under time reversal, then $\vec{A} \cdot \vec{P} = 0$.

Proof.

$\vec{A} \cdot \vec{P}$ is invariant under translations and rotations. Therefore, $\vec{A} \cdot \vec{P}\Big|_{\Gamma^*}$ is a function of $|\vec{P}|, \hat{P} \cdot \vec{s}$, i.e.

$$\vec{A} \cdot \vec{P}\Big|_{\Gamma^*} = F(|\vec{P}|, \hat{P} \cdot \vec{s}). \tag{8}$$

Under time reversal, we have $|\vec{P}| \circ T_u = |\vec{P} \circ T_u| = |-\vec{P}| = |\vec{P}|$ and $(\hat{P} \cdot \vec{s}) \circ T_u = \hat{P} \cdot \vec{s}$,

Proof: step 1

Lemma

If $\vec{A}\colon \Gamma \to \mathbb{R}^3$ *is* C^1*, invariant under translations, a vector under spatial rotations and invariant under time reversal, then* $\vec{A} \cdot \vec{P} = 0$.

Proof.

$\vec{A} \cdot \vec{P}$ is invariant under translations and rotations. Therefore, $\vec{A} \cdot \vec{P}\Big|_{\Gamma^*}$ is a function of $|\vec{P}|, \hat{P} \cdot \vec{s}$, i.e.

$$\vec{A} \cdot \vec{P}\Big|_{\Gamma^*} = F(|\vec{P}|, \hat{P} \cdot \vec{s}). \tag{8}$$

Under time reversal, we have $|\vec{P}| \circ T_u = |\vec{P} \circ T_u| = |-\vec{P}| = |\vec{P}|$ and $(\hat{P} \cdot \vec{s}) \circ T_u = \hat{P} \cdot \vec{s}$, implying $F(|\vec{P}|, \hat{P} \cdot \vec{s}) \circ T_u = F(|\vec{P}|, \hat{P} \cdot \vec{s})$.

Proof: step 1

Lemma

If $\vec{A}: \Gamma \to \mathbb{R}^3$ is C^1, invariant under translations, a vector under spatial rotations and invariant under time reversal, then $\vec{A} \cdot \vec{P} = 0$.

Proof.

$\vec{A} \cdot \vec{P}$ is invariant under translations and rotations. Therefore, $\vec{A} \cdot \vec{P}\Big|_{\Gamma^*}$ is a function of $|\vec{P}|, \hat{P} \cdot \vec{s}$, i.e.

$$\vec{A} \cdot \vec{P}\Big|_{\Gamma^*} = F(|\vec{P}|, \hat{P} \cdot \vec{s}). \tag{8}$$

Under time reversal, we have $|\vec{P}| \circ T_u = |\vec{P} \circ T_u| = |-\vec{P}| = |\vec{P}|$ and $(\hat{P} \cdot \vec{s}) \circ T_u = \hat{P} \cdot \vec{s}$, implying $F(|\vec{P}|, \hat{P} \cdot \vec{s}) \circ T_u = F(|\vec{P}|, \hat{P} \cdot \vec{s})$. On the other hand, we have $(\vec{A} \cdot \vec{P}) \circ T_u = -\vec{A} \cdot \vec{P}$.

Proof: step 1

Lemma

If $\vec{A}\colon \Gamma \to \mathbb{R}^3$ *is* C^1*, invariant under translations, a vector under spatial rotations and invariant under time reversal, then* $\vec{A} \cdot \vec{P} = 0$.

Proof.

$\vec{A} \cdot \vec{P}$ is invariant under translations and rotations. Therefore, $\vec{A} \cdot \vec{P}\big|_{\Gamma^*}$ is a function of $|\vec{P}|, \hat{P} \cdot \vec{s}$, i.e.

$$\vec{A} \cdot \vec{P}\Big|_{\Gamma^*} = F(|\vec{P}|, \hat{P} \cdot \vec{s}). \tag{8}$$

Under time reversal, we have $|\vec{P}| \circ T_u = |\vec{P} \circ T_u| = |-\vec{P}| = |\vec{P}|$ and $(\hat{P} \cdot \vec{s}) \circ T_u = \hat{P} \cdot \vec{s}$, implying $F(|\vec{P}|, \hat{P} \cdot \vec{s}) \circ T_u = F(|\vec{P}|, \hat{P} \cdot \vec{s})$. On the other hand, we have $(\vec{A} \cdot \vec{P}) \circ T_u = -\vec{A} \cdot \vec{P}$. We thus obtain $\vec{A} \cdot \vec{P}\big|_{\Gamma^*} = 0$, and continuity implies $\vec{A} \cdot \vec{P} = 0$. □

Proof: finishing for zero spin

- The P_a form a complete involutive set on Γ.

Proof: finishing for zero spin

- The P_a form a complete involutive set on Γ.
- Thus $\vec{d} = \vec{X} - \vec{x}$, being translation invariant, must be a function of $\vec{P}$.

Proof: finishing for zero spin

- The P_a form a complete involutive set on Γ.
- Thus $\vec{d} = \vec{X} - \vec{x}$, being translation invariant, must be a function of $\vec{P}$.
- Being a vector under rotations, it must be of the form

$$\vec{d}(\vec{P}) = F(|\vec{P}|)\vec{P}. \tag{9}$$

Proof: finishing for zero spin

- The P_a form a complete involutive set on Γ.
- Thus $\vec{d} = \vec{X} - \vec{x}$, being translation invariant, must be a function of $\vec{P}$.
- Being a vector under rotations, it must be of the form

$$\vec{d}(\vec{P}) = F(|\vec{P}|)\vec{P}. \tag{9}$$

- But $\vec{d} \cdot \vec{P} = 0$, so $\vec{d} = 0$.

Proof: finishing for zero spin

- The P_a form a complete involutive set on Γ.
- Thus $\vec{d} = \vec{X} - \vec{x}$, being translation invariant, must be a function of $\vec{P}$.
- Being a vector under rotations, it must be of the form

$$\vec{d}(\vec{P}) = F(|\vec{P}|)\vec{P}. \tag{9}$$

- But $\vec{d} \cdot \vec{P} = 0$, so $\vec{d} = 0$.
- Note that we didn't use $\{X^a, X^b\} = 0$!

Proof: step 2 for non-zero spin

Lemma

If $\vec{A}\colon \Gamma \to \mathbb{R}^3$ *is* C^1*, invariant under translations, a vector under spatial rotations and satisfies* $\vec{A} \cdot \vec{P} = 0$*, it is of the form*

$$\vec{A} = B\hat{P} \times \vec{s} + C\hat{P} \times (\hat{P} \times \vec{s}) \tag{10}$$

on $\Gamma^* \setminus \{\vec{s} \parallel \hat{P}\}$*, where* B *and* C *are* C^1 *functions of* $|\vec{P}|$ *and* $\hat{P} \cdot \vec{s}$*.*

Proof: step 2 for non-zero spin

Lemma

If $\vec{A}\colon \Gamma \to \mathbb{R}^3$ *is* C^1*, invariant under translations, a vector under spatial rotations and satisfies* $\vec{A} \cdot \vec{P} = 0$*, it is of the form*

$$\vec{A} = B\hat{P} \times \vec{s} + C\hat{P} \times (\hat{P} \times \vec{s}) \tag{10}$$

on $\Gamma^* \setminus \{\vec{s} \parallel \hat{P}\}$*, where* B *and* C *are* C^1 *functions of* $|\vec{P}|$ *and* $\hat{P} \cdot \vec{s}$*.*

Proof.

The $\mathbb{R}^3$-valued functions $\hat{P}, \hat{P} \times \vec{s}, \hat{P} \times (\hat{P} \times \vec{s})$ form an orthogonal basis of $\mathbb{R}^3$.

Proof: step 2 for non-zero spin

Lemma

If $\vec{A}\colon \Gamma \to \mathbb{R}^3$ is C^1, invariant under translations, a vector under spatial rotations and satisfies $\vec{A} \cdot \vec{P} = 0$, it is of the form

$$\vec{A} = B\hat{P} \times \vec{s} + C\hat{P} \times (\hat{P} \times \vec{s}) \tag{10}$$

on $\Gamma^* \setminus \{\vec{s} \parallel \hat{P}\}$, where B and C are C^1 functions of $|\vec{P}|$ and $\hat{P} \cdot \vec{s}$.

Proof.

The $\mathbb{R}^3$-valued functions $\hat{P}, \hat{P} \times \vec{s}, \hat{P} \times (\hat{P} \times \vec{s})$ form an orthogonal basis of $\mathbb{R}^3$. Expanding $\vec{A}$ in this basis gives (10) with

$$B = \frac{\vec{A} \cdot (\hat{P} \times \vec{s})}{|\hat{P} \times \vec{s}|}, \quad C = \frac{\vec{A} \cdot (\hat{P} \times (\hat{P} \times \vec{s}))}{|\hat{P} \times (\hat{P} \times \vec{s})|}. \tag{11}$$

Proof: step 2 for non-zero spin

Lemma

If $\vec{A}\colon \Gamma \to \mathbb{R}^3$ is C^1, invariant under translations, a vector under spatial rotations and satisfies $\vec{A}\cdot\vec{P} = 0$, it is of the form

$$\vec{A} = B\hat{P}\times\vec{s} + C\hat{P}\times(\hat{P}\times\vec{s}) \tag{10}$$

on $\Gamma^* \setminus \{\vec{s} \parallel \hat{P}\}$, where B and C are C^1 functions of $|\vec{P}|$ and $\hat{P}\cdot\vec{s}$.

Proof.

The $\mathbb{R}^3$-valued functions $\hat{P}, \hat{P}\times\vec{s}, \hat{P}\times(\hat{P}\times\vec{s})$ form an orthogonal basis of $\mathbb{R}^3$. Expanding $\vec{A}$ in this basis gives (10) with

$$B = \frac{\vec{A}\cdot(\hat{P}\times\vec{s})}{|\hat{P}\times\vec{s}|},\ C = \frac{\vec{A}\cdot(\hat{P}\times(\hat{P}\times\vec{s}))}{|\hat{P}\times(\hat{P}\times\vec{s})|}. \tag{11}$$

Thus, B, C are invariant under translations and rotations.

Proof: step 2 for non-zero spin

Lemma

If $\vec{A}\colon \Gamma \to \mathbb{R}^3$ is C^1, invariant under translations, a vector under spatial rotations and satisfies $\vec{A} \cdot \vec{P} = 0$, it is of the form

$$\vec{A} = B\hat{P} \times \vec{s} + C\hat{P} \times (\hat{P} \times \vec{s}) \tag{10}$$

on $\Gamma^* \setminus \{\vec{s} \parallel \hat{P}\}$, where B and C are C^1 functions of $|\vec{P}|$ and $\hat{P} \cdot \vec{s}$.

Proof.

The $\mathbb{R}^3$-valued functions $\hat{P}, \hat{P} \times \vec{s}, \hat{P} \times (\hat{P} \times \vec{s})$ form an orthogonal basis of $\mathbb{R}^3$. Expanding $\vec{A}$ in this basis gives (10) with

$$B = \frac{\vec{A} \cdot (\hat{P} \times \vec{s})}{|\hat{P} \times \vec{s}|}, \; C = \frac{\vec{A} \cdot (\hat{P} \times (\hat{P} \times \vec{s}))}{|\hat{P} \times (\hat{P} \times \vec{s})|}. \tag{11}$$

Thus, B, C are invariant under translations and rotations. Therefore, they are functions of $|\vec{P}|$ and $\hat{P} \cdot \vec{s}$. □

Proof: step 3 for non-zero spin

- $\vec{d} \cdot \vec{P} = 0$ means $\vec{X} \cdot \vec{P} = \vec{x} \cdot \vec{P}$.

Proof: step 3 for non-zero spin

- $\vec{d} \cdot \vec{P} = 0$ means $\vec{X} \cdot \vec{P} = \vec{x} \cdot \vec{P}$.
- Since we assume $\{X^a, P_b\} = \delta^a_b$ and $\{X^a, X^b\} = 0$, this implies

$$\{X^a, \vec{x} \cdot \vec{P}\} = \{X^a, \vec{X} \cdot \vec{P}\} = X^a. \tag{12}$$

Proof: step 3 for non-zero spin

- $\vec{d} \cdot \vec{P} = 0$ means $\vec{X} \cdot \vec{P} = \vec{x} \cdot \vec{P}$.
- Since we assume $\{X^a, P_b\} = \delta^a_b$ and $\{X^a, X^b\} = 0$, this implies

$$\{X^a, \vec{x} \cdot \vec{P}\} = \{X^a, \vec{X} \cdot \vec{P}\} = X^a. \tag{12}$$

- Combining this with $\{x^a, \vec{x} \cdot \vec{P}\} = x^a$, we obtain

$$\{d^a, \vec{x} \cdot \vec{P}\} = d^a. \tag{13}$$

Proof: step 3 for non-zero spin

- $\vec{d} \cdot \vec{P} = 0$ means $\vec{X} \cdot \vec{P} = \vec{x} \cdot \vec{P}$.
- Since we assume $\{X^a, P_b\} = \delta^a_b$ and $\{X^a, X^b\} = 0$, this implies

$$\{X^a, \vec{x} \cdot \vec{P}\} = \{X^a, \vec{X} \cdot \vec{P}\} = X^a. \tag{12}$$

- Combining this with $\{x^a, \vec{x} \cdot \vec{P}\} = x^a$, we obtain

$$\{d^a, \vec{x} \cdot \vec{P}\} = d^a. \tag{13}$$

- On the other hand, for any function F of $\vec{P}$ and $\vec{s}$, we have:

$$\{F(\vec{P}, \vec{s}), \vec{x} \cdot \vec{P}\} = -\frac{\partial F(\vec{P}, \vec{s})}{\partial P_a} P_a \tag{14}$$

Proof: step 3 for non-zero spin

- $\vec{d} \cdot \vec{P} = 0$ means $\vec{X} \cdot \vec{P} = \vec{x} \cdot \vec{P}$.
- Since we assume $\{X^a, P_b\} = \delta^a_b$ and $\{X^a, X^b\} = 0$, this implies

$$\{X^a, \vec{x} \cdot \vec{P}\} = \{X^a, \vec{X} \cdot \vec{P}\} = X^a. \tag{12}$$

- Combining this with $\{x^a, \vec{x} \cdot \vec{P}\} = x^a$, we obtain

$$\{d^a, \vec{x} \cdot \vec{P}\} = d^a. \tag{13}$$

- On the other hand, for any function F of $\vec{P}$ and $\vec{s}$, we have:

$$\{F(\vec{P}, \vec{s}), \vec{x} \cdot \vec{P}\} = -\frac{\partial F(\vec{P}, \vec{s})}{\partial P_a} P_a = -|\vec{P}| \left. \frac{\partial F}{\partial |\vec{P}|} \right|_{\hat{P}, \vec{s} = \text{const.}} \tag{14}$$

Proof: step 3 for non-zero spin

- $\vec{d} \cdot \vec{P} = 0$ means $\vec{X} \cdot \vec{P} = \vec{x} \cdot \vec{P}$.
- Since we assume $\{X^a, P_b\} = \delta^a_b$ and $\{X^a, X^b\} = 0$, this implies
$$\{X^a, \vec{x} \cdot \vec{P}\} = \{X^a, \vec{X} \cdot \vec{P}\} = X^a. \tag{12}$$
- Combining this with $\{x^a, \vec{x} \cdot \vec{P}\} = x^a$, we obtain
$$\{d^a, \vec{x} \cdot \vec{P}\} = d^a. \tag{13}$$
- On the other hand, for any function F of $\vec{P}$ and $\vec{s}$, we have:
$$\{F(\vec{P}, \vec{s}), \vec{x} \cdot \vec{P}\} = -\frac{\partial F(\vec{P}, \vec{s})}{\partial P_a} P_a = -|\vec{P}| \left. \frac{\partial F}{\partial |\vec{P}|} \right|_{\hat{P}, \vec{s}=\text{const.}} \tag{14}$$
- Thus:
$$\vec{d} = -|\vec{P}| \left. \frac{\partial \vec{d}}{\partial |\vec{P}|} \right|_{\hat{P}=\text{const.}, \vec{s}=\text{const.}} \tag{15}$$

Proof: finishing for non-zero spin

- We now have:

$$\vec{d} = B\hat{P} \times \vec{s} + C\hat{P} \times (\hat{P} \times \vec{s}) \text{ on } \Gamma^* \setminus \{\vec{s} \parallel \hat{P}\} \tag{10}$$

$$\vec{d} = -|\vec{P}| \left. \frac{\partial \vec{d}}{\partial |\vec{P}|} \right|_{\hat{P}=\text{const.},\vec{s}=\text{const.}} \tag{15}$$

Proof: finishing for non-zero spin

- We now have:

$$\vec{d} = B\hat{P} \times \vec{s} + C\hat{P} \times (\hat{P} \times \vec{s}) \text{ on } \Gamma^* \setminus \{\vec{s} \parallel \hat{P}\} \tag{10}$$

$$\vec{d} = -|\vec{P}| \left. \frac{\partial \vec{d}}{\partial |\vec{P}|} \right|_{\hat{P}=\text{const.},\vec{s}=\text{const.}} \tag{15}$$

- Combining those:

$$B(|\vec{P}|, \hat{P} \cdot \vec{s}) = -|\vec{P}| \frac{\partial B(|\vec{P}|, \hat{P} \cdot \vec{s})}{\partial |\vec{P}|}, \tag{16}$$

same for C.

Proof: finishing for non-zero spin

- We now have:

$$\vec{d} = B\hat{P} \times \vec{s} + C\hat{P} \times (\hat{P} \times \vec{s}) \text{ on } \Gamma^* \setminus \{\vec{s} \parallel \hat{P}\} \tag{10}$$

$$\vec{d} = -|\vec{P}| \left. \frac{\partial \vec{d}}{\partial |\vec{P}|} \right|_{\hat{P}=\text{const.},\vec{s}=\text{const.}} \tag{15}$$

- Combining those:

$$B(|\vec{P}|, \hat{P} \cdot \vec{s}) = -|\vec{P}| \frac{\partial B(|\vec{P}|, \hat{P} \cdot \vec{s})}{\partial |\vec{P}|}, \tag{16}$$

  same for C.
- This means $B, C \propto |\vec{P}|^{-1}$.

Proof: finishing for non-zero spin

- We now have:

$$\vec{d} = B\hat{P} \times \vec{s} + C\hat{P} \times (\hat{P} \times \vec{s}) \text{ on } \Gamma^* \setminus \{\vec{s} \parallel \hat{P}\} \quad (10)$$

$$\vec{d} = -|\vec{P}| \left. \frac{\partial \vec{d}}{\partial |\vec{P}|} \right|_{\hat{P}=\text{const.},\vec{s}=\text{const.}} \quad (15)$$

- Combining those:

$$B(|\vec{P}|, \hat{P} \cdot \vec{s}) = -|\vec{P}| \frac{\partial B(|\vec{P}|, \hat{P} \cdot \vec{s})}{\partial |\vec{P}|}, \quad (16)$$

  same for C.
- This means $B, C \propto |\vec{P}|^{-1}$.
- Continuity on all of Γ implies $B = C = 0$.

Proof: finishing for non-zero spin

- We now have:

$$\vec{d} = B\hat{P} \times \vec{s} + C\hat{P} \times (\hat{P} \times \vec{s}) \text{ on } \Gamma^* \setminus \{\vec{s} \parallel \hat{P}\} \tag{10}$$

$$\vec{d} = -|\vec{P}| \left. \frac{\partial \vec{d}}{\partial |\vec{P}|} \right|_{\hat{P}=\text{const.},\vec{s}=\text{const.}} \tag{15}$$

- Combining those:

$$B(|\vec{P}|, \hat{P} \cdot \vec{s}) = -|\vec{P}| \frac{\partial B(|\vec{P}|, \hat{P} \cdot \vec{s})}{\partial |\vec{P}|}, \tag{16}$$

  same for C.
- This means $B, C \propto |\vec{P}|^{-1}$.
- Continuity on all of Γ implies $B = C = 0$. □

Conclusion

- Newton–Wigner theorem has a direct classical analogue
- Some ‘problems’ / unintuitive features of Newton–Wigner localisation are not of quantum-mechanical, but of special-relativistic origin (e.g. supposed non-covariance)

Conclusion

- Newton–Wigner theorem has a direct classical analogue
- Some 'problems' / unintuitive features of Newton–Wigner localisation are not of quantum-mechanical, but of special-relativistic origin (e.g. supposed non-covariance)

Many thanks for your attention!

Leibniz Universität Hannover

CRC 1227
Designed Quantum States of Matter

Appendix: Details

Complete involutive sets

Let (M, ω) be a symplectic manifold of dimension $2n$.

Definition

A *complete involutive set* on (M, ω) is a set of $n = \frac{\dim M}{2}$ smooth functions $f_1, \ldots, f_n \in C^\infty(M, \mathbb{R})$ satisfying:

1. $\{f_i, f_j\} = 0$ *(involutivity)*
2. The Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_n}$ are pointwise linearly independent. *(completeness)*

Complete involutive sets

Let $f_1, \ldots, f_n$ form a complete involutive set on (M, ω).

Theorem (Liouville–Arnold)

Locally, there are functions $g^1, \ldots, g^n$ such that $\omega = \mathrm{d}f_i \wedge \mathrm{d}g^i$.

Back

Complete involutive sets

Let $f_1, \dots, f_n$ form a complete involutive set on (M, ω).

Theorem (Liouville–Arnold)

Locally, there are functions $g^1, \dots, g^n$ such that $\omega = \mathrm{d}f_i \wedge \mathrm{d}g^i$.

Corollary

If $h \in C^\infty(M, \mathbb{R})$ Poisson-commutes with all of the f_i, then h is a function of the f_i.

Back