Geometric descriptions of (post-)Newtonian modified gravity

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Outline

PKS: *Teleparallel Newton–Cartan gravity*, `arXiv:2211.11796`, CQG **40**, 105008 (2023);
PKS: *The classification of general affine connections in Newton–Cartan geometry: Towards metric-affine Newton–Cartan gravity*, `arXiv:2403.15460`;
PKS, AL von Blanckenburg: *The Newtonian limit of orthonormal frames in metric theories of gravity*, `arXiv:2410.01800`

Motivation

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Motivation 1: Newtonian gravity

- $\mathbb{R} \times \mathbb{R}^3$ with coordinates $(t, \vec{x})$
- Newtonian field equation: gravitational potential ϕ solves the Poisson equation

$$\Delta\phi = 4\pi G\rho, \tag{1}$$

  where ρ is the mass density
- Test particles move according to the force equation

$$\ddot{x}^a(t) = -\partial_a \phi(t, \vec{x}(t)) \tag{2}$$

- Example: point mass $\rho(t, \vec{x}) = M\delta(\vec{x} - \vec{x}_0)$, boundary condition $\phi \xrightarrow{|\vec{x}| \to \infty} 0 \implies$

$$\text{potential } \phi(t, \vec{x}) = -\frac{GM}{|\vec{x} - \vec{x}_0|}, \quad \text{acceleration } -\vec{\nabla}\phi(t, \vec{x}) = -\frac{GM}{|\vec{x} - \vec{x}_0|^2} \frac{\vec{x} - \vec{x}_0}{|\vec{x} - \vec{x}_0|} \tag{3}$$

Motivation 2: general relativity

Modern theory of gravity: general relativity (GR)

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Axioms for general relativity

1. Spacetime is a Lorentzian manifold (M, g),
2. ideal clocks and rods measure proper time and lengths as defined by g,
3. the equations of motion for matter are formulated in the spacetime geometry given by g and its Levi–Civita connection $\widetilde{\nabla}$,
4. the field equation

$$\widetilde{R}_{\mu\nu} - \frac{1}{2}\widetilde{R}g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{4}$$

holds, where $T_{\mu\nu}$ is the energy–momentum tensor of matter.

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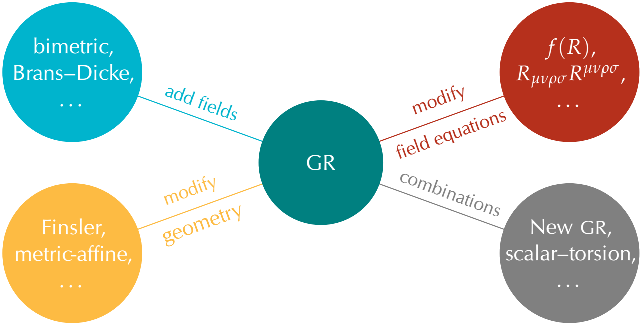
Spacetime tells matter how to move; matter tells spacetime how to curve.

'Gravity' is not a force (= causing non-inertial motion), but an effect of curved spacetime geometry.

Motivation 3: modified gravity

- GR agrees very precisely with observations: solar system, spacecraft missions, astrophysics, gravitational waves, cosmology, . . .
- However, challenges remain: quantisation, dark matter, cosmological constant / dark energy, Hubble tension, . . .
- Possible solution: modified theories of gravity



Motivation 4: Newtonian limits

- Newtonian limit: GR $\xrightarrow[\text{small velocities}]{\text{weak fields}}$ Newtonian gravity
  - Textbook formulation: linearised gravity
  - Coordinate dependent

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- Geometric formulation: $\text{GR} \xrightarrow{c \to \infty}$ *Newton–Cartan gravity* (Cartan, Friedrichs, Trautman, Dixon, Künzle, Ehlers, ...)
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Goal

Develop similar geometric descriptions of the (post-)Newtonian behaviour of modified theories of gravity, for a proper understanding.

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- This talk: *teleparallel equivalent of GR* (TEGR) as concrete example

Standard Newton–Cartan gravity

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Galilei manifolds

- *Galilei manifold*: M with $\dim M = 4$, nowhere vanishing *clock form* $\tau \in \Omega^1(M)$, symmetric *space metric* $h \in \Gamma(TM \otimes TM)$ of signature $(0{+}{+}{+})$, $\tau_\mu h^{\mu\nu} = 0$

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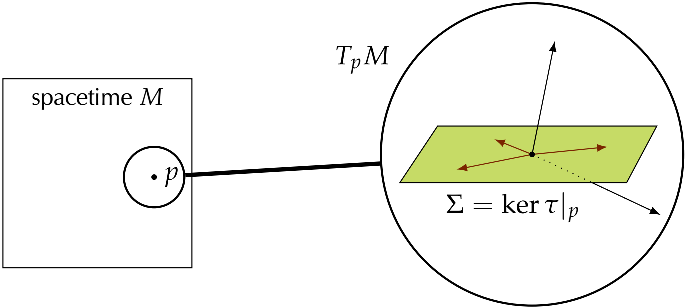

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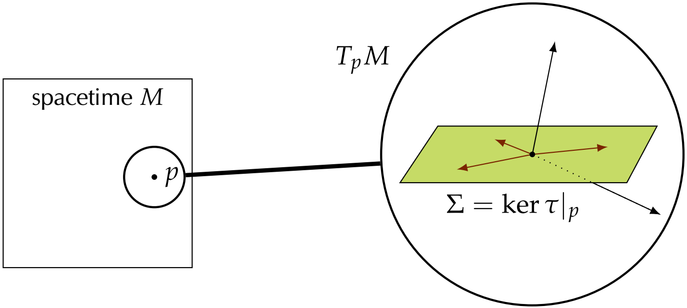

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- $\ker \tau = \{$*spacelike* vectors$\}$, others: *timelike*
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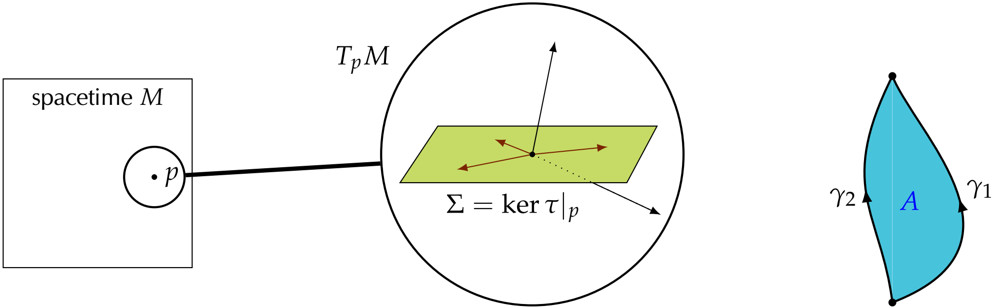
- This talk: mostly assume $\mathrm{d}\tau = 0 \rightsquigarrow$ absolute time: $\int_{\gamma_1} \tau - \int_{\gamma_2} \tau = \int_{\partial A} \tau = \int_A \mathrm{d}\tau$



Galilei frames

- Homogeneous Galilei group:

$$\mathrm{Gal} = \mathrm{O}(3) \ltimes \mathbb{R}^3 \tag{5}$$

- $\mathrm{Gal} \subset \mathrm{GL}(4)$ via

$$\mathrm{Gal} \ni (R, k) \mapsto \begin{pmatrix} 1 & 0 \\ k & R \end{pmatrix} \in \mathrm{GL}(4) \tag{6}$$

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- *Galilei frame* for (M, τ, h): local frame $(\mathrm{e}_A) = (\mathrm{e}_t = v, \mathrm{e}_a)$ s.t. $\tau(v) = 1,\ h = \delta^{ab} \mathrm{e}_a \otimes \mathrm{e}_b$

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- Pointwise basis change matrices between Galilei frames: precisely elements of Gal
- $\rightsquigarrow$ *Galilei frame bundle* $G(M) \xrightarrow{\pi} M$, reduction of the structure group of the linear frame bundle $F(M)$ to Gal

Galilei connections

- *Galilei connection*: connection $\boldsymbol{\omega}$ on $G(M)$
- on M: covariant derivative ∇ on TM with $\nabla\tau = 0 = \nabla h$, or local connection form $(\omega^a{}_b, \varpi^a)$ valued in $\mathfrak{gal} = \mathfrak{so}(3) \oplus \mathbb{R}^3$

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Theorem (Künzle, Bernal–Sánchez, Bekaert–Morand)

With respect to a choice of unit timelike vector field v, any Galilei connection has the form

$$\Gamma^{\rho}_{\mu\nu} = \overset{v}{\Gamma}{}^{\rho}_{\mu\nu} - \frac{1}{2} v^{\rho} (\mathrm{d}\tau)_{\mu\nu} + \frac{1}{2} T^{\lambda}{}_{\mu\nu} - T_{(\mu\nu)}{}^{\rho} + \tau_{(\mu} \Omega_{\nu)}{}^{\rho} \tag{7}$$

in terms of its torsion T, which satisfies $\tau(T(\cdot,\cdot)) = \mathrm{d}\tau$, and its Newton–Coriolis form

$$\Omega = \delta_{ab} \varpi^a \wedge \mathrm{e}^b. \tag{8}$$

Conversely, for arbitrary T and Ω (arbitrary 2-form), (7) defines a Galilei connection.

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Conversely, for arbitrary T and Ω (arbitrary 2-form), (7) defines a Galilei connection.

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- *Newtonian connection*: $T = 0$ and $\mathrm{d}\Omega = 0$ ($\Longleftrightarrow {R^\mu}_\nu{}^\rho{}_\sigma = {R^\rho}_\sigma{}^\mu{}_\nu$)

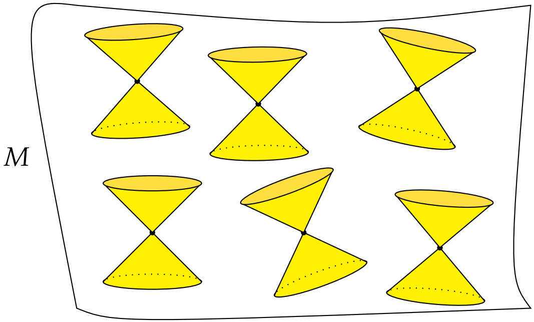
Newton–Cartan gravity

Axioms for Newton–Cartan gravity

1. Spacetime is a Galilei manifold (M, τ, h) with absolute time, endowed with a Newtonian connection $\widetilde{\omega}$,
2. ideal clocks measure time as defined by τ, ideal rods measure lengths as defined by h,
3. free test particles move on timelike geodesics of $\widetilde{\omega}$, i.e. timelike curves γ solving
$$\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = 0, \tag{9}$$
4. the field equation
$$\widetilde{R}_{\mu\nu} = 4\pi G\,\rho\,\tau_\mu \tau_\nu \tag{10}$$
holds, where ρ is the mass density.

- Arises as formal $c \to \infty$ limit of GR
- Coordinate formulation of Newtonian gravity can be recovered (up to possibility of non-absolute rotation)

Newtonian limits of Lorentzian manifolds



- Lorentzian manifold (M, g)

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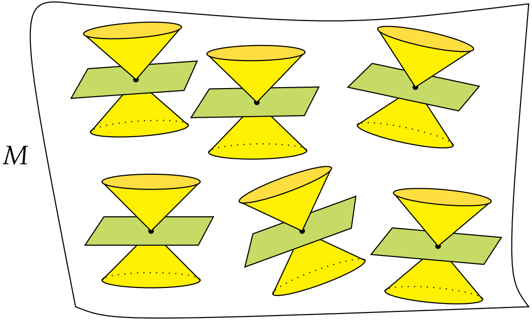

- Lorentzian manifold (M, g), reference state of motion

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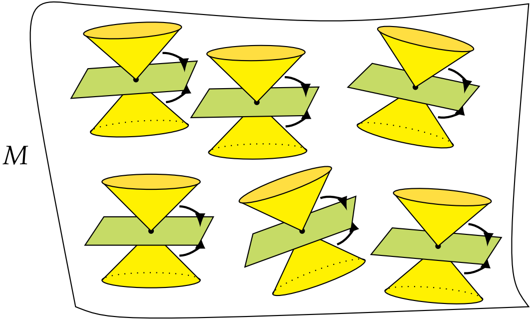

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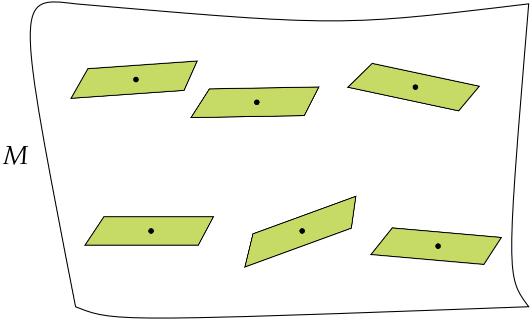

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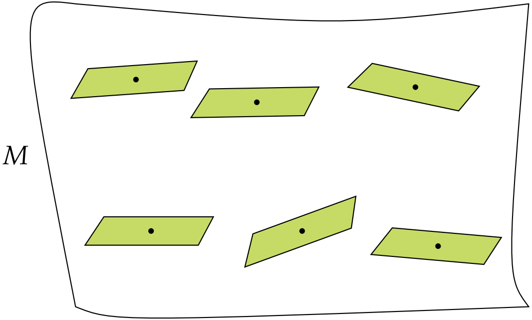

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- Allow tensor fields with values in formal Laurent series field $\mathbb{R}((c^{-1}))$
- Expansion of Lorentzian objects in c^{-1} $\rightsquigarrow$ Lorentzian geometry as formal deformation of Galilei geometry

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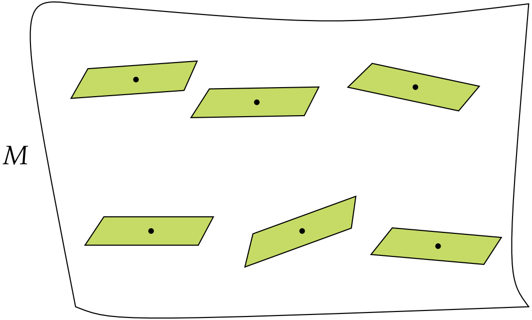

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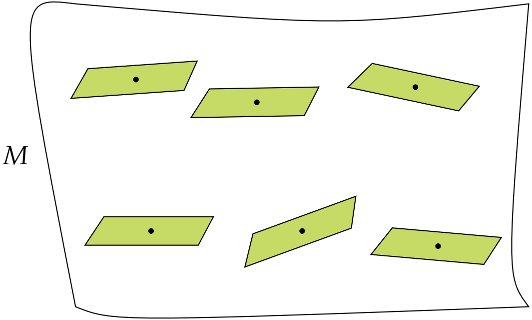

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- Structure group contracted from Lorentz to (hom.) Galilei

Newton–Cartan gravity from GR

Theorem (Künzle)

Let (M, g) be a Lorentzian manifold, and assume that the metric and inverse metric have expansions $g = -c^2\tau \otimes \tau + \mathrm{O}(c^0)$, $g^{-1} = h + \mathrm{O}(c^{-2})$ with τ nowhere vanishing. Then:

- *(M, τ, h) is a Galilei manifold.*

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- *The trace-reversed Einstein equation for (M, g) goes over to the Newton–Cartan field equation for $(M, \tau, h, \tilde{\nabla})$.*

▸ Details

Example: the Newtonian limit of Schwarzschild

$$g = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 \mathrm{d}t^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2) \tag{11a}$$

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- h induces standard Euclidean metric on spatial hypersurfaces
- $\tilde{\nabla}$ gives Newtonian trajectories

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\begin{aligned}
g^{-1} &= -\left(1-\frac{2GM}{c^2 r}\right)^{-1} c^{-2} \partial_t \otimes \partial_t + \left(1-\frac{2GM}{c^2 r}\right) \partial_r \otimes \partial_r + r^{-2}(\partial_\theta \otimes \partial_\theta + \sin^{-2}\theta\, \partial_\varphi \otimes \partial_\varphi) \\
&= \partial_r \otimes \partial_r + r^{-2}(\partial_\theta \otimes \partial_\theta + \sin^{-2}\theta\, \partial_\varphi \otimes \partial_\varphi) + \mathrm{O}(c^{-2}) \qquad (11b)
\end{aligned}
$$

- $\tau = \mathrm{d}t$—absolute time!
- h induces standard Euclidean metric on spatial hypersurfaces
- $\widetilde{\nabla}$ gives Newtonian trajectories
- Same limit for Kerr (assuming regularity outside horizon for all values of c)

Teleparallelisation of Newton–Cartan gravity

PKS: *Teleparallel Newton–Cartan gravity*, `arXiv:2211.11796`, CQG **40**, 105008 (2023);
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- Equivalent formulations of GR:
  - *Teleparallel Equivalent of GR* (TEGR), *Symmetric Teleparallel Equivalent of GR* (STEGR)
  - Different starting points for further modifications

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  - $\implies$ Fixing $\overset{L}{\nabla}$, the Levi-Civita connection $\overset{L}{\tilde{\nabla}}$ may be expressed in terms of g, $\overset{L}{\nabla}$, and $\overset{L}{T}$
  - $\rightsquigarrow$ Riemannian curvature $\overset{L}{\tilde{R}}$ expressible purely in terms of $g, \overset{L}{\nabla}, \overset{L}{T}, \overset{L}{R}$
  - Choose $\overset{L}{\nabla}$ *flat*[1], i.e. $\overset{L}{R} = 0$ $\rightsquigarrow$ Reformulation of GR in terms of $g, \overset{L}{\nabla}, \overset{L}{T}$

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- Analogue for NC gravity?
  - Problem: Galilei connection ω not uniquely determined by T!

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Extending Galilei structures with Bargmann forms

- Bargmann group $\mathrm{Barg} = \mathrm{Gal} \ltimes (\mathbb{R}^4 \times \mathrm{U}(1))$ ▸ Details

'Globalised' construction from:
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- Pulled back to M: *extended coframe* $(\mathrm{e}^t = \tau, \mathrm{e}^a, \mathrm{i}a)$
- Transformation under local boost of frame $v \mapsto v - k^a \mathrm{e}_a$:

$$a \mapsto a + k_a \mathrm{e}^a + \frac{1}{2}|k|^2 \tau \qquad (12)$$

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- For $0 = \mathrm{d}\tau = T^t$: *unique* Galilei connection $\tilde{\omega}$ with vanishing extended torsion
- This enables 'teleparallelisation' of Newton–Cartan gravity!
- For general $\boldsymbol{\omega}$: *Newton–Cartan contortion*

$$\Gamma^{\rho}_{\mu\nu} - \widetilde{\Gamma}^{\rho}_{\mu\nu} = \frac{1}{2} T^{\rho}{}_{\mu\nu} - T_{(\mu\nu)}{}^{\rho} + \tau_{(\mu} f_{\nu)}{}^{\rho} =: K^{\rho}{}_{\mu\nu} \tag{14}$$

Teleparallel Newton–Cartan gravity

Axioms for teleparallel Newton–Cartan gravity

1. Spacetime is a Galilei manifold (M, τ, h) with absolute time, endowed with a Bargmann form and a flat Galilei connection ω,
2. ideal clocks measure time as defined by τ, ideal rods measure lengths as defined by h,
3. free test particles move on timelike curves γ solving

$$(\nabla_{\dot{\gamma}}\dot{\gamma})^\rho = K^\rho{}_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu, \tag{15}$$

4. the field equation

$$-\nabla_\rho K^\rho{}_{\mu\nu} + \nabla_\mu K^\rho{}_{\rho\nu} - K^\rho{}_{\sigma\nu}T^\sigma{}_{\rho\mu} + K^\rho{}_{\rho\sigma}K^\sigma{}_{\mu\nu} - K^\rho{}_{\mu\sigma}K^\sigma{}_{\rho\nu} = 4\pi G\,\rho\,\tau_\mu\,\tau_\nu \tag{16}$$

holds, where ρ is the mass density.

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- LHS of (16) is $\widetilde{R}_{\mu\nu}$ $\rightsquigarrow$ equivalent to standard NC gravity

Teleparallel NC: relation to other theories

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Recall: Expansion of Lorentzian objects in c^{-1} $\rightsquigarrow$ Lorentzian geometry as formal deformation of Galilei geometry

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(M, g) *Lorentzian manifold. Assume an* ON *coframe and frame expand as follows* $(\tau|_p \neq 0)$*:*

$$\mathrm{E}^0 = c\tau + c^{-1}a + \mathrm{O}(c^{-3}), \quad \mathrm{E}^a = \mathrm{e}^a + \mathrm{O}(c^{-2}), \quad \mathrm{E}_0 = c^{-1}v + \mathrm{O}(c^{-3}), \quad \mathrm{E}_a = \mathrm{e}_a + \mathrm{O}(c^{-2})$$

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- *$\mathrm{d}\tau = 0$: Field equation of TEGR $\rightsquigarrow$ trace-reverse $\xrightarrow{c \to \infty}$ teleparallel NC field eq.!*

Convergence of ON frames

The limiting behaviour of the frame is actually 'automatic', given convergence of the metric:

Theorem (PKS, AL von Blanckenburg)

Assume that a one-parameter family of Lorentzian metrics converges in the Newtonian limit to a Galilei structure τ, h. Then any family of ON frames for these metrics converges pointwise to a Galilei frame, in the manner assumed earlier, given that the two obvious necessary conditions are satisfied:

1. *the frame's boost velocity with respect to some fixed reference observer needs to converge, and*
2. *the spatial frame must not rotate indefinitely as the limit is approached.*

Details

PKS, AL von Blanckenburg: *The Newtonian limit of orthonormal frames in metric theories of gravity*, arXiv:2410.01800

Recovering Newtonian gravity from teleparallel NC

- ‘Gauge fix’ to $T^a{}_{bc} = 0$ Details

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Thus we recovered standard Newtonian gravity from teleparallel Newton–Cartan gravity!

Outlook and conclusion

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Beyond Newton: coordinate-free post-Newtonian expansions

- Recently: extension of Newton–Cartan limit of GR to post-Newtonian expansions
- Higher-order terms in c^{-1} expansion of Lorentzian geometry
- Expansion of Einstein–Hilbert action
  $\rightsquigarrow$ variational formulation of post-Newton–Cartan expansion of GR
- Needs generalisation to $\mathrm{d}\tau \neq 0$ ('torsional Newton–Cartan', TNC) $\rightsquigarrow$ captures effects of strong gravity

D Van den Bleeken: *Torsional Newton–Cartan gravity from the large c expansion of general relativity*, arXiv:1703.03459, CQG **34**, 185004 (2017);
D Hansen, J Hartong, NA Obers: *Action Principle for Newtonian Gravity*, arXiv:1807.04765, PRL **122**, 061106 (2019);
D Hansen, J Hartong, NA Obers: *Gravity between Newton and Einstein*, arXiv:1904.05706, IJMPD **28**, 1944010 (2019);
D Hansen, J Hartong, NA Obers: *Non-relativistic gravity and its coupling to matter*, arXiv:2001.10277, JHEP **2020**, 145 (2020)

Beyond TEGR: different modified theories

- My work on TEGR (PKS 2023): first geometric Newtonian limit of modified gravity

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Including non-metricity: classification of general connections

Theorem (PKS)

On a Galilei manifold (M, τ, h), a choice of unit timelike vector field $v \in \Gamma(TM)$, $\tau(v) = 1$ induces an affine isomorphism between the affine space of affine connections on M and the vector space

$$\Gamma(\ker\tau \otimes \textstyle\bigwedge^2 T^*M) \oplus \Gamma(T^*M \otimes T^*M) \oplus \Gamma(T^*M \otimes \textstyle\bigvee^2 \ker\tau) \oplus \Gamma(\textstyle\bigwedge^2 T^*M)\,,$$

mapping a connection ∇ to the field with components

$$(P^\rho_\lambda T^\lambda{}_{\mu\nu}\,, \hat{Q}_{\mu\nu}\,, Q_\rho{}^{\kappa\lambda} P^\mu_\kappa P^\nu_\lambda\,, \Omega_{\mu\nu})$$

in terms of the torsion T and 'non-metricities' $\hat{Q} = \nabla\tau$, $Q = \nabla h$ of ∇, the projection $P = \mathrm{id}_{TM} - v \otimes \tau$ associated with v, and the Newton–Coriolis form Ω of ∇ with respect to v.

PKS: *The classification of general affine connections in Newton–Cartan geometry: Towards metric-affine Newton–Cartan gravity*, `arXiv:2403.15460`

Conclusion

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- Proper understanding of (post-)Newtonian gravity needs geometric formulations!
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Outlook

- Coordinate-free variational formulation of post-Newtonian (S)TEGR
- Analogous descriptions of post-Newtonian behaviour of true modified theories

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Summary

- Proper understanding of (post-)Newtonian gravity needs geometric formulations!
- Bargmann form formalism for Galilei geometry $\rightsquigarrow$ teleparallel formulation of NC gravity, geometrising the $c \to \infty$ limit of TEGR

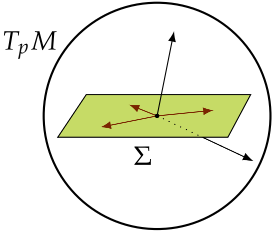
Outlook

- Coordinate-free variational formulation of post-Newtonian (S)TEGR
- Analogous descriptions of post-Newtonian behaviour of true modified theories

Many thanks for your attention!

Appendix: details

Spatial measurements



- *Space metric* $h \in T_pM \otimes T_pM$, $h^{\mu\nu} = h^{\nu\mu}$, signature $(0{+}{+}{+})$, $h^{\mu\nu}\tau_\nu = 0$

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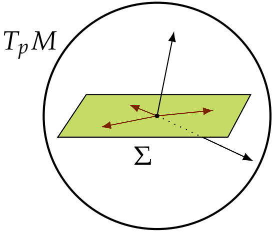

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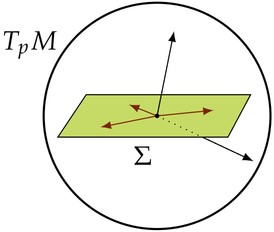

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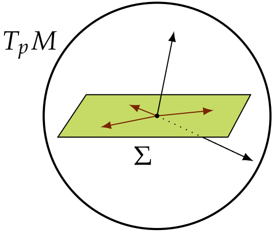

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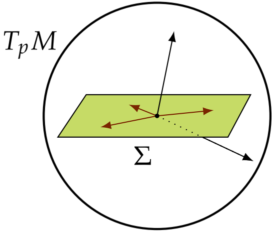

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- Conversely:

$$ h \colon T_p^*M \twoheadrightarrow \Sigma^* \xrightarrow[\cong]{{}^{(\Sigma)}h^{-1}} \Sigma \hookrightarrow T_pM \tag{20} $$

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Details on the limit GR → Newton–Cartan

- $g = -c^2 \tau \otimes \tau + \mathrm{O}(c^0)$, $g^{-1} = h + \mathrm{O}(c^{-2})$

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- Christoffel symbols of g:

$$\overset{\mathrm{L}}{\tilde{\Gamma}}{}^{\rho}_{\mu\nu} = -c^2 h^{\rho\sigma} \tau_{(\mu} (\mathrm{d}\tau)_{\nu)\sigma} + O(c^0) \tag{21}$$

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- Assumed expansion of energy–momentum tensor:

$$T_{\mu\nu} = \hat{\rho} c^2 V_\mu V_\nu + c\Pi_\mu V_\nu + V_\mu c\Pi_\nu + \Sigma_{\mu\nu} \tag{22a}$$

with $V_\mu := \tau_\mu / \sqrt{-g^{-1}(\tau, \tau)}$, and $\hat{\rho}, \Pi, \Sigma$ of order c^0. Energy density $\to$ mass density: $\hat{\rho} = \rho + \mathrm{O}(c^{-1})$.

$$\implies T_{\mu\nu} = \rho c^4 \tau_\mu \tau_\nu + c^3 (\dots) \tau_\mu \tau_\nu + c^2 [\tau_\mu (\dots) + (\dots) \tau_\nu] + \mathrm{O}(c^1) \tag{22b}$$

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The Barmann group and algebra

- Homomorphism $\rho\colon \mathrm{Gal} \to \mathrm{Aut}(\mathbb{R}^4 \times \mathrm{U}(1))$ given by

$$\rho_{(R,k)}(y^A, \exp(\mathrm{i}\varphi)) = \left(y^t, R^a{}_b y^b + y^t k^a, \exp(\mathrm{i}(\varphi + \tfrac{1}{2}|k|^2 y^t + k_a R^a{}_b y^b))\right) \tag{23}$$

- Bargmann group $\mathrm{Barg} = \mathrm{Gal} \ltimes_\rho (\mathbb{R}^4 \times \mathrm{U}(1))$
- Induced homomorphism $\dot\rho'\colon \mathfrak{gal} \to \mathrm{Der}(\mathbb{R}^4 \oplus \mathfrak{u}(1))$ given by

$$\dot\rho'_{(X,k)}(y^A, \mathrm{i}\varphi) = \left(((X,k)y)^A, k_a y^a\right) \tag{24}$$

- Bargmann algebra $\mathfrak{barg} = \mathfrak{gal} \oplus_{\dot\rho'} (\mathbb{R}^4 \oplus \mathfrak{u}(1))$
- $\mathfrak{barg}$ is the essentially unique 1D non-trivial central extension of $\mathfrak{igal} = \mathfrak{gal} \oplus \mathbb{R}^4$

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Gauge-theoretic interpretation of Bargmann forms

- Extend Galilei frame bundle to Barg-bundle $B(M) = G(M) \times_{\mathrm{Gal}} \mathrm{Barg}$
- Connection on $B(M) \leftrightarrow$ Galilei connection $\boldsymbol{\omega}$ + tensorial form $(\boldsymbol{\theta}, \mathrm{i}\boldsymbol{a}) \in \Omega^1_{\mathrm{Gal}}(G(M), \mathbb{R}^4 \oplus \mathfrak{u}(1))$
- Generalises classical case of affine connections:
  Connection on affine frame bundle $A(M) \leftrightarrow$ connection on linear frame bundle $F(M)$ $+$ tensorial form $\boldsymbol{\theta} \in \Omega^1_{\mathrm{GL}(4)}(F(M), \mathbb{R}^4)$

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Details on convergence of ON frames

Theorem (PKS, AL von Blanckenburg)

Write $\lambda = c^{-2}$. Let $\overset{\lambda}{g}$ be a λ-dependent family of Lorentzian metrics on a real vector space V, $\dim V = n+1$, that satisfies $\lambda \overset{\lambda}{g} \xrightarrow{\lambda\to 0} -\tau\otimes\tau$ and $\overset{\lambda}{g}{}^{-1} \xrightarrow{\lambda\to 0} h$ such that τ, h define a Galilei structure on V. Let $(\overset{\lambda}{\mathrm{E}}_A)$ be a family of ON bases for $\overset{\lambda}{g}$, such that its boost velocity with respect to a fixed reference observer converges.
Then we have $\sqrt{\lambda}\overset{\lambda}{\mathrm{E}}{}^0 \xrightarrow{\lambda\to 0} \pm\tau$, and $\mathrm{e}_t := \pm\lim_{\lambda\to 0}\frac{1}{\sqrt{\lambda}}\overset{\lambda}{\mathrm{E}}_0$ is unit timelike with respect to τ.
Extending e_t to a Galilei basis $(\mathrm{e}_t, \mathrm{e}_a)$, there is a family $\overset{\lambda}{A} = (\overset{\lambda}{A}{}^a{}_b)^n_{a,b=1}$ of matrices in $\mathrm{O}(n)$ such that $(\overset{\lambda}{A}{}^{-1})^a{}_b \overset{\lambda}{\mathrm{E}}{}^b \xrightarrow{\lambda\to 0} \mathrm{e}^a$ and $\overset{\lambda}{A}{}^a{}_b \overset{\lambda}{\mathrm{E}}_a \xrightarrow{\lambda\to 0} \mathrm{e}_b$.
If the assumed limits are differentiable as $\lambda\to 0$, then so are the concluded ones.

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PKS, AL von Blanckenburg: *The Newtonian limit of orthonormal frames in metric theories of gravity*, arXiv:2410.01800

Gauge fixing the purely spatial torsion

- Purely spatial part of field equation: $\widetilde{R}_{ab} = 0 \overset{\text{3D}}{\implies}$ spatial metric flat
- $\implies$ We may assume

$$T^a{}_{bc} = 0 \tag{25}$$

consistently with flatness

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Trautman's 'absolute rotation' condition

- Usual NC: To recover Newtonian gravity, assume $\widetilde{R}^{ab}{}_{\mu\nu} = 0$
- $\iff$ $\exists$ rigid, non-rotating frames
- Such frames in teleparallel NC:

$$\varpi_{(ab)} = T_{(ab)t} \tag{26a}$$

$$\varpi_{[ab]} = \frac{1}{2} f_{ab} \tag{26b}$$

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