The classification of general affine connections in Galilei geometry

Towards metric-affine Newton–Cartan gravity

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Outline

PKS: *The classification of general affine connections in Newton–Cartan geometry: towards metric-affine Newton–Cartan gravity*, `arXiv:2403.15460`, CQG **42**, 015010 (2025)

Motivation

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Motivation 1: gravity as geometry

- Main lesson[1] of general relativity (GR): gravity is an effect of spacetime geometry.

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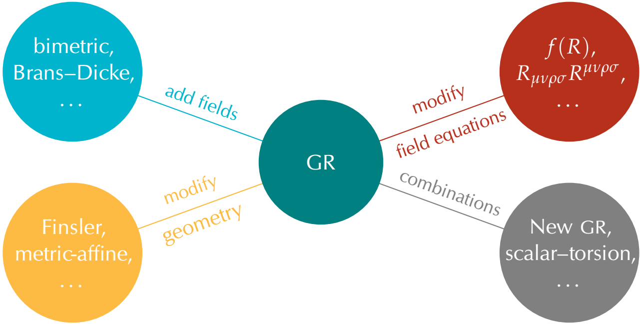

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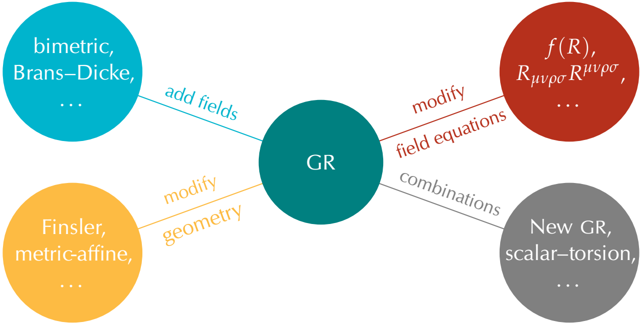

- Understanding an aspect of gravity means understanding it geometrically!

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  - ...
- Starting points for further modifications
- Generic metric-affine theory
  - g and ∇ (a priori) independent
  - T, Q (possibly) dynamical

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Goal

Develop a classification of affine connections in the context of Galilei geometry, as a foundation for a similar geometric understanding of the (post-)Newtonian behaviour of metric-affine gravity.

Galilei geometry

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Galilei manifolds

- *Galilei manifold*: M with $\dim M = n$, nowhere-vanishing *clock form* $\tau \in \Gamma(T^*M)$, symmetric *space metric* $h \in \Gamma(\bigvee^2 TM)$ of signature $(0+\cdots+)$, $\tau_\mu h^{\mu\nu} = 0$

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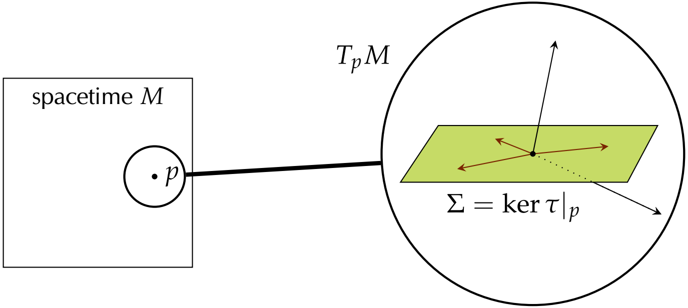

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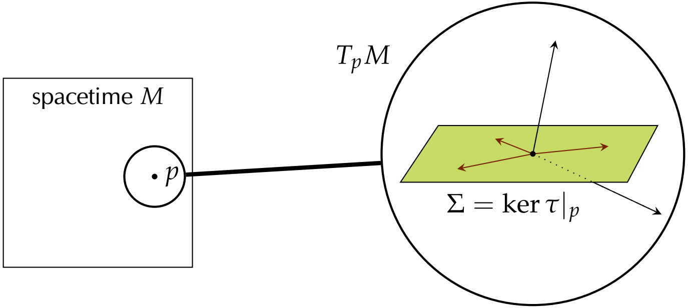

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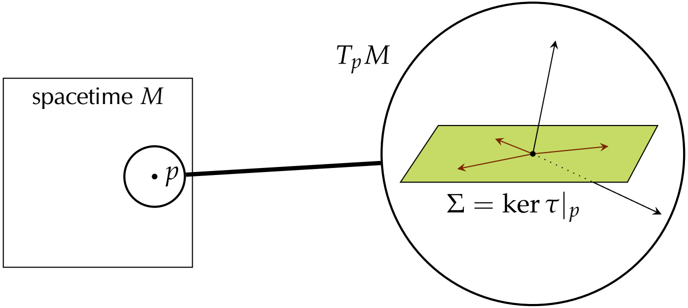
- $\int_\gamma \tau =$ elapsed time along γ, h defines bundle metric on $\ker\tau$ Details on bundle metric
- Lorentzian geometry $\xrightarrow{c\to\infty}$ Galilei geometry Details on Newtonian limit



Unit timelike vector fields

Unit timelike vector field: $v \in \Gamma(TM)$ with $\tau(v) = 1$
Induced structure:

- Decomposition of tangent and cotangent bundles

$$TM = \mathrm{span}\{v\} \oplus \ker\tau\ , \tag{1a}$$

$$T^*M = \mathrm{span}\{\tau\} \oplus \ker v \tag{1b}$$

- Spatial projector $P = \mathrm{id}_{TM} - v \otimes \tau \colon TM \to TM$
- *Covariant space metric* $h_{\mu\nu}$ defined by

$$h_{\mu\nu} v^\nu = 0\ , \quad h_{\mu\nu} h^{\nu\rho} = P_\mu^\rho \tag{2}$$

Convention

We raise indices with $h^{\mu\nu}$ and lower them with $h_{\mu\nu}$.

Classifying connections

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The pseudo-Riemannian case

Theorem

On a pseudo-Riemannian manifold (M, g), the coefficients of an affine connection ∇ can be expressed in terms of its torsion T and non-metricity $Q = \nabla g$, as

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) + \frac{1}{2} T^{\rho}{}_{\mu\nu} - T_{(\mu\nu)}{}^{\rho} + \frac{1}{2} Q^{\rho}{}_{\mu\nu} - Q_{(\mu\nu)}{}^{\lambda} . \quad (3)$$

Conversely, for arbitrary T and Q, this defines a connection with torsion T and non-metricity Q.

Put differently: There is an affine isomorphism between the affine space of affine connections on M and the vector space

$$\Gamma(TM \otimes \textstyle\bigwedge^2 T^*M) \oplus \Gamma(T^*M \otimes \textstyle\bigvee^2 T^*M) , \quad (4)$$

mapping a connection ∇ to its torsion and non-metricity.

'∇ = Levi-Civita + contortion(T) + disformation(Q)'

The Galilei case: differences / intricacies

Consider a Galilei manifold (M, τ, h) and an affine connection ∇ on it.

- Several 'metrics' $\tau, h \implies$ several 'non-metricities'
- τ, h related $\implies$ non-metricities not independent
- Differential form τ determining geometry $\implies$ restriction of torsion
- 'Metrics' are degenerate $\implies$ non-metricity and torsion not sufficient!

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  Additional freedom encoded by *Newton–Coriolis form* of ∇ with respect to any unit timelike vector field v,

$$\Omega_{\mu\nu} := 2(\nabla_{[\mu} v^{\rho}) h_{\nu]\rho} \,. \tag{5}$$

The Galilei case: classification result concretely

Fix a Galilei manifold (M, τ, h) and a unit timelike vector field v on it.

Theorem (PKS)

The coefficients of any affine connection ∇ on M can be expressed in terms of its torsion T, non-metricities $\hat{Q} := \nabla\tau$, $Q := \nabla h$, and Newton–Coriolis form Ω with respect to v, as

$$\Gamma^{\rho}_{\mu\nu} = v^{\rho}\partial_{\mu}\tau_{\nu} + \frac{1}{2}h^{\rho\sigma}(\partial_{\mu}h_{\nu\sigma} + \partial_{\nu}h_{\mu\sigma} - \partial_{\sigma}h_{\mu\nu}) + \frac{1}{2}P^{\rho}_{\lambda}T^{\lambda}{}_{\mu\nu} - T_{(\mu\nu)}{}^{\rho} + \tau_{(\mu}\Omega_{\nu)}{}^{\rho} - \frac{1}{2}Q^{\rho}{}_{\mu\nu} + P^{\rho}_{\lambda}Q_{(\mu\nu)}{}^{\lambda} - v^{\rho}\hat{Q}_{\mu\nu}\ . \tag{6}$$

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The torsion and non-metricities satisfy

$$\tau_{\mu}Q_{\rho}{}^{\mu\nu} = -\hat{Q}_{\rho\mu}h^{\mu\nu} \,, \quad \tau_{\rho}T^{\rho}{}_{\mu\nu} = (\mathrm{d}\tau)_{\mu\nu} - 2\hat{Q}_{[\mu\nu]} \,. \tag{7}$$

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Conversely, for arbitrary T, $\hat{Q}$, Q satisfying (7) and an arbitrary 2-form $\Omega \in \Gamma(\bigwedge^2 T^*M)$, (6) defines a connection ∇ with torsion T that satisfies $\nabla\tau = \hat{Q}$, $\nabla h = Q$, and has Newton–Coriolis form Ω with respect to v.

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This generalises the well-known metric-compatible case (Künzle 1972, Bernal–Sánchez 2003, Bekaert–Morand 2016).

The Galilei case: relations between non-metricities and torsion

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  - τ-non-metricity $\hat{Q}$ freely specifiable, determining the restricted parts of T and Q

The Galilei case: classification result abstractly

Theorem (PKS)

On a Galilei manifold (M, τ, h), a choice of unit timelike vector field $v \in \Gamma(TM)$, $\tau(v) = 1$ induces an affine isomorphism between the affine space of affine connections on M and the vector space

$$\Gamma(\ker\tau \otimes \bigwedge^2 T^*M) \oplus \Gamma(T^*M \otimes T^*M) \oplus \Gamma(T^*M \otimes \bigvee^2 \ker\tau) \oplus \Gamma(\bigwedge^2 T^*M)\,,$$

mapping a connection ∇ to the field with components

$$(P^\rho_\lambda T^\lambda{}_{\mu\nu}\,, \hat{Q}_{\mu\nu}\,, Q_\rho{}^{\kappa\lambda} P^\mu_\kappa P^\nu_\lambda\,, \Omega_{\mu\nu})$$

in terms of the torsion T and 'non-metricities' $\hat{Q} = \nabla\tau$, $Q = \nabla h$ of ∇, the projection $P = \mathrm{id}_{TM} - v \otimes \tau$ associated with v, and the Newton–Coriolis form Ω of ∇ with respect to v.

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The explicit formula (6) provides the inverse of this isomorphism.

The principal bundle perspective

- Galilei manifold $(M, \tau, h) \rightsquigarrow$ Galilei frame bundle $G(M)$ of adapted frames (principal bundle, with structure group the hom. Galilei group)
- Well-known: affine connections ∇ on M compatible with τ, h $\longleftrightarrow$ principal connections ω on $G(M)$
- Generalisation: general affine connections ∇ on M $\longleftrightarrow$ specific equivariant $\mathfrak{gl}(n+1)$-valued forms ω on $G(M)$
  - Image in $\mathfrak{gl}(n+1)/\mathfrak{gal}$ encodes non-metricities

'Gauge theoretic' perspective

Galilei geometry with general connection as 'gauging' of the affine algebra, split into translation, hom. Galilei, and non-metricity parts.

Details

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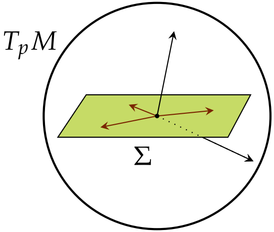
Many thanks for your attention!

Appendix: details

Spatial measurements



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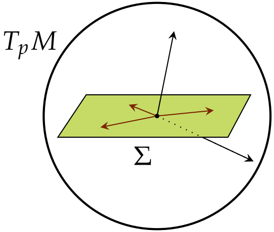

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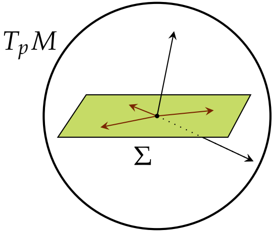

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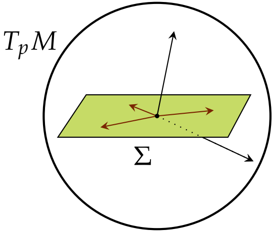
- Hyperplane $\Sigma = \ker \tau|_p = \operatorname{im} h$ of spacelike vectors
- h defines scalar product ${}^{(\Sigma)}h$ on Σ, i.e. ${}^{(\Sigma)}h \in \Sigma^* \otimes \Sigma^*$:

$$ {}^{(\Sigma)}h^{-1} \colon \Sigma^* \cong T_p^*M/\operatorname{span}\{\tau|_p\} = T_p^*M/\ker h \xrightarrow[\cong]{\tilde{h}} \operatorname{im} h = \Sigma \tag{8} $$

Back

Spatial measurements



- *Space metric* $h \in T_pM \otimes T_pM$, $h^{\mu\nu} = h^{\nu\mu}$, signature $(0{+}{+}{+})$, $h^{\mu\nu}\tau_\nu = 0$
- Hyperplane $\Sigma = \ker \tau|_p = \operatorname{im} h$ of spacelike vectors
- h defines scalar product ${}^{(\Sigma)}h$ on Σ, i.e. ${}^{(\Sigma)}h \in \Sigma^* \otimes \Sigma^*$:

$$ {}^{(\Sigma)}h^{-1} \colon \Sigma^* \cong T_p^*M/\operatorname{span}\{\tau|_p\} = T_p^*M/\ker h \xrightarrow[\cong]{\tilde{h}} \operatorname{im} h = \Sigma \tag{8} $$

- $\rightsquigarrow$ *spatial distances* defined between infinitesimally close simultaneous events

Back

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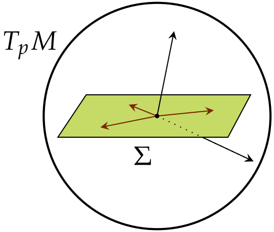

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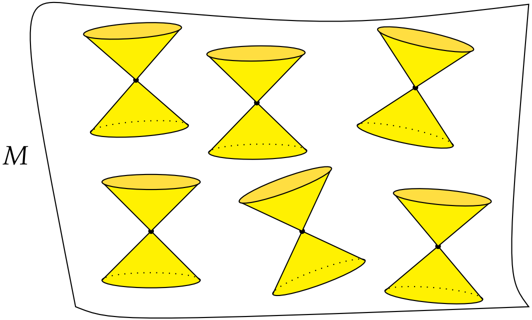
- Conversely:

$$ h \colon T_p^*M \twoheadrightarrow \Sigma^* \xrightarrow[\cong]{{}^{(\Sigma)}h^{-1}} \Sigma \hookrightarrow T_pM \tag{9} $$

Back

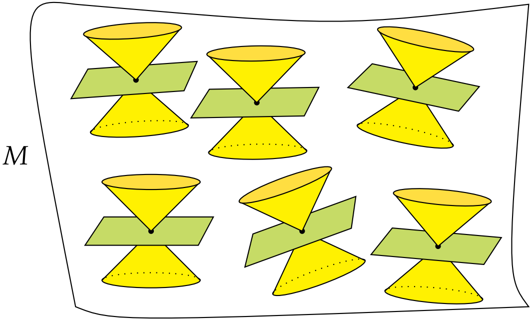
Newtonian limits of Lorentzian manifolds



- Lorentzian manifold (M, g)

Back

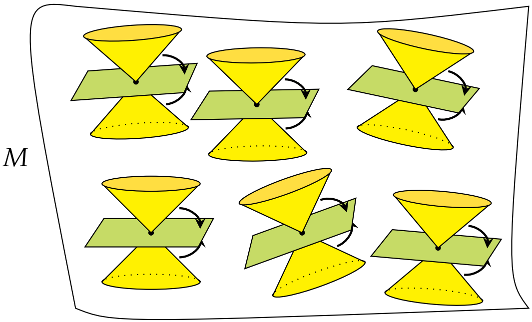
Newtonian limits of Lorentzian manifolds



- Lorentzian manifold (M, g), reference state of motion

Back

Newtonian limits of Lorentzian manifolds



- Lorentzian manifold (M, g), reference state of motion $\implies$ Newtonian limit '$c \to \infty$': local lightcones flatten to Newtonian causal structure

Back

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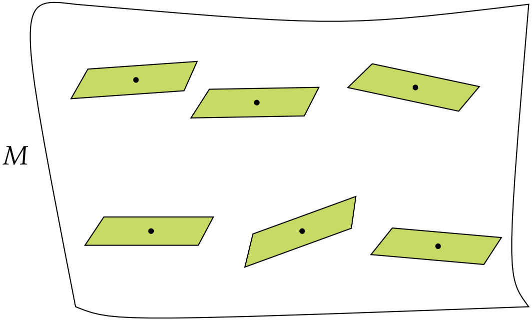

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Back

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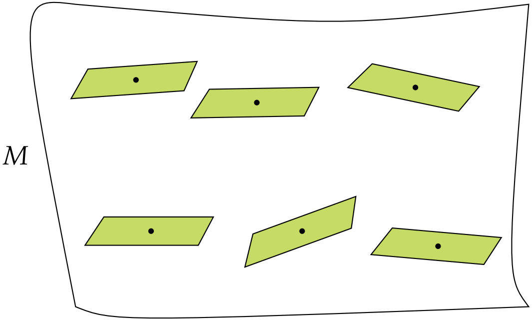

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- Allow tensor fields with values in formal Laurent series field $\mathbb{R}((c^{-1}))$
- Expansion of Lorentzian objects in c^{-1} $\rightsquigarrow$ Lorentzian geometry as formal deformation of Galilei geometry

Back

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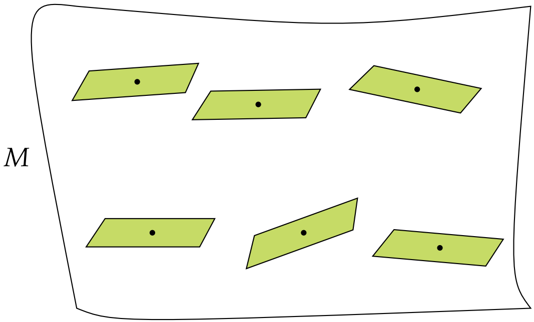

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- Allow tensor fields with values in formal Laurent series field $\mathbb{R}((c^{-1}))$
- Expansion of Lorentzian objects in c^{-1} $\rightsquigarrow$ Lorentzian geometry as formal deformation of Galilei geometry
- Reference: choice of τ! Assume expansion $g = -c^2\tau \otimes \tau + \mathrm{O}(c^0)$, $g^{-1} = h + \mathrm{O}(c^{-2})$

Back

Galilei frames

- Homogeneous Galilei group:
$$\mathrm{Gal} = \mathrm{O}(n) \ltimes \mathbb{R}^n \tag{10}$$
- $\mathrm{Gal} \subset \mathrm{GL}(n+1)$ via
$$\mathrm{Gal} \ni (R,k) \mapsto \begin{pmatrix} 1 & 0 \\ k & R \end{pmatrix} \in \mathrm{GL}(n+1) \tag{11}$$

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- $\rightsquigarrow$ *Galilei frame bundle* $G(M) \xrightarrow{\pi} M$, reduction of the structure group of the linear frame bundle $F(M)$ to Gal
- Newtonian limit: Lorentzian ON frames converge to Galilei frames (Details), structure group contracted from Lorentz to Galilei

Classification result: the principal bundle perspective

Theorem (PKS)

Let (M, τ, h) be a Galilei manifold, $G(M)$ its Galilei frame bundle and $F(M)$ the general linear frame bundle of M. An affine connection ∇ on M corresponds to a connection form $\hat{\omega}$ on $F(M)$, or equivalently—taking the pullback of $\hat{\omega}$ along the inclusion $G(M) \hookrightarrow F(M)$—to a form $\omega \in \Gamma(T^*G(M) \otimes \mathfrak{gl}(n+1))$ satisfying the following:

1. *ω is equivariant with respect to the representation $\rho\colon \mathrm{Gal} \to \mathrm{Aut}(\mathfrak{gl}(n+1))$ that arises by restricting the adjoint representation of $\mathrm{GL}(n+1)$ to Gal.*
2. *Evaluated on the fundamental vector field $\widetilde{(X,k)} \in \Gamma(TG(M))$ of an element $(X,k) \in \mathfrak{gal}$, the form ω yields the matrix corresponding to (X,k) in $\mathfrak{gl}(n+1)$.*

Composing ω with the projection $\mathfrak{gl}(n+1) \to \mathfrak{gl}(n+1)/\mathfrak{gal}$, we obtain a form $\omega + \mathfrak{gal}$ on $G(M)$ with values in $\mathfrak{gl}(n+1)/\mathfrak{gal}$ that is tensorial with respect to the quotient representation $\mathrm{Gal} \to \mathrm{GL}(\mathfrak{gl}(n+1)/\mathfrak{gal})$. This encodes the non-metricities $\nabla\tau$ and ∇h of the affine connection ∇.

(For the explicit form of the non-metricities, see the paper.)

Back

Convergence of ON frames

The limiting behaviour of the frame is actually 'automatic', given convergence of the metric:

Theorem (PKS, AL von Blanckenburg)

Assume that a one-parameter family of Lorentzian metrics converges in the Newtonian limit to a Galilei structure τ, h. Then any family of ON frames for these metrics converges pointwise to a Galilei frame, in the manner assumed earlier, given that the two obvious necessary conditions are satisfied:

1. *the frame's boost velocity with respect to some fixed reference observer needs to converge, and*
2. *the spatial frame must not rotate indefinitely as the limit is approached.*

▸ Even more details

◂ Back

PKS, AL von Blanckenburg: *The Newtonian limit of orthonormal frames in metric theories of gravity*, `arXiv:2410.01800`

Details on convergence of ON frames

Theorem (PKS, AL von Blanckenburg)

Write $\lambda = c^{-2}$. Let $\overset{\lambda}{g}$ be a λ-dependent family of Lorentzian metrics on a real vector space V, $\dim V = n+1$, that satisfies $\lambda \overset{\lambda}{g} \xrightarrow{\lambda \to 0} -\tau \otimes \tau$ and $\overset{\lambda}{g}{}^{-1} \xrightarrow{\lambda \to 0} h$ such that τ, h define a Galilei structure on V. Let $(\overset{\lambda}{\mathrm{E}}_A)$ be a family of ON bases for $\overset{\lambda}{g}$, such that its boost velocity with respect to a fixed reference observer converges.

Then we have $\sqrt{\lambda}\overset{\lambda}{\mathrm{E}}{}^0 \xrightarrow{\lambda \to 0} \pm\tau$, and $\mathrm{e}_t := \pm \lim_{\lambda \to 0} \frac{1}{\sqrt{\lambda}} \overset{\lambda}{\mathrm{E}}_0$ is unit timelike with respect to τ.

Extending e_t to a Galilei basis $(\mathrm{e}_t, \mathrm{e}_a)$, there is a family $\overset{\lambda}{A} = (\overset{\lambda}{A}{}^a{}_b)^n_{a,b=1}$ of matrices in $\mathrm{O}(n)$ such that $(\overset{\lambda}{A}{}^{-1})^a{}_b \overset{\lambda}{\mathrm{E}}{}^b \xrightarrow{\lambda \to 0} \mathrm{e}^a$ and $\overset{\lambda}{A}{}^a{}_b \overset{\lambda}{\mathrm{E}}_a \xrightarrow{\lambda \to 0} \mathrm{e}_b$.

If the assumed limits are differentiable as $\lambda \to 0$, then so are the concluded ones.

Back

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