The Schrödinger field, and Newtonian limits of matter actions

[E22] *Local boost invariance of the Schrödinger action*
Show by explicit computation that the action for the Schrödinger field on a Galilei spacetime, as introduced in construction 5.12 in the lecture, is invariant under local Galilei boosts.

[E23] *Bargmann symmetry of the free Schrödinger equation*
From the construction of the Bargmann structure framework and the invariance of the action, we know that the Schrödinger equation on any Galilei manifold (M, τ, h) with absolute time and with Bargmann structure $\boldsymbol{a}$,

$$\mathrm{i}\hbar v^\mu \mathrm{D}_\mu \Psi + \frac{\mathrm{i}\hbar}{2}(\nabla_\mu v^\mu)\Psi = -\frac{\hbar^2}{2m} h^{\mu\nu} \mathrm{D}_\mu \mathrm{D}_\nu \Psi \tag{1}$$

(discussed in construction 5.12 in the lecture), is symmetric under all of the following transformations:

(i) combined $\mathrm{U}(1)$ gauge transformations of $\boldsymbol{a}$ and Ψ,

(ii) Milne boosts / local Galilei boosts of v and a,

(iii) pushforward of τ, h, v, a, Ψ by any diffeomorphism $\varphi\colon M \to M$.[1]

In this exercise we will use this to show that the free Schrödinger equation

$$\mathrm{i}\hbar \partial_t \Psi = -\frac{\hbar^2}{2m} \delta^{ab} \partial_a \partial_b \Psi \tag{2}$$

on $\mathbb{R}^{n+1}$ is symmetric under a specific action of (a slightly modified version of) the Bargmann group. We consider the Newtonian manifold (M, τ, h, ∇) where $M = \mathbb{R}^{n+1}$ with affine coordinates (t, x^a), $\tau = \mathrm{d}t$, $h = \delta^{ab}\partial_a \otimes \partial_b$, and ∇ is the standard linear connection whose connection coefficients with respect to the coordinates (t, x^a) vanish.

(a) Show that any diffeomorphism $\varphi\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ under whose pushforward action τ, h, ∇ are invariant[2], i.e. $\varphi_*\tau = \tau$, $\varphi_* h = h$, $\varphi_*\nabla = \nabla$ is given by the standard action of an element (R, k, y) of the inhomogeneous Galilei group acting on $\mathbb{R}^{n+1}$, i.e.

$$\varphi(t, x^a) = (t + y^t, R^a{}_b x^b + t k^a + y^a). \tag{3}$$

Hint: From the condition $\varphi^*\nabla = \nabla$, you know that φ has to be an affine map of $\mathbb{R}^{n+1}$, i.e. that φ can be written as multiplication with a constant matrix followed

[1] One easily checks that if a is the local representative of a Bargmann structure on the Galilei manifold (M, τ, h) with respect to the Galilei frame (e_A), then for any diffeomorphism $\varphi\colon M \to N$ the form $\varphi_* a$ will be the local representative of a Bargmann structure on the Galilei manifold $(N, \varphi_*\tau, \varphi_* h)$ with respect to the Galilei frame $(\varphi_*\mathrm{e}_A)$, so we can really transport Bargmann structures by pushforward by pushing forward their local representatives.

[2] Note that invariance of ∇ will not be used in the rest of the exercise, it is just a nice way to obtain Galilei transformations.

by translation by a constant vector. The remaining conditions then restrict this to an element of IGal.

(b) We now consider the unit timelike vector field $v = \partial_t$. To this, we first apply a Milne boost / local Galilei boost parametrised by a spacelike vector field $\tilde{k}$ on $\mathbb{R}^{n+1}$, and second pushforward by a Galilei transformation diffeomorphism $\varphi_{(R,k,y)}$ as in part (a). Show that for v to be invariant under this combined action, i.e. for

$$(\varphi_{(R,k,y)})_*(v - \tilde{k}) = v, \tag{4}$$

we need $\tilde{k} = (R^{-1})^a{}_b k^b \partial_a$.

(c) We consider the Bargmann structure $\boldsymbol{a}$ on (M, τ, h) whose local representative with respect to $v = \partial_t$ vanishes, $a = 0$. As in part (b), to the local representative of our Bargmann structure we apply a local Galilei boost by $(R^{-1})^a{}_b k^b \partial_a$, followed by pushforward by $\varphi_{(R,k,y)}$. This defines an action of the inhomogeneous Galilei group on Bargmann structures on our (M, τ, h).[3] Compute the local representative of the resulting new Bargmann structure.

(d) Before the action of part (c), we now also apply a U(1) gauge transformation parametrised by $e^{i\chi}$ to our original Bargmann structure with $a = 0$. For which functions χ is the resulting Bargmann structure $\tilde{\boldsymbol{a}}$ equal to the original one, i.e. again locally given by $\tilde{a} = 0$?

Hint: Evaluate the condition $0 = \tilde{a} = \varphi_*(\ldots)$ *in the equivalent form* $\cdots = 0$. *You should obtain* $\chi = -k_a R^a{}_b x^b + \frac{1}{2}|k|^2 t + \textit{const}$.

(e) Combining all the previous parts, we see that the structure $(\tau, h, v, \boldsymbol{a})$ is invariant under the combined action (pushforward by $\varphi_{(R,k,y)}$) $\circ$ (Milne boost by $R^{-1}k$) $\circ$ (U(1) transformation by χ as in part (d)). How does this combination act on a Schrödinger field, understood geometrically as in construction 5.12 from the lecture? Show that this defines an action on Schrödinger fields by a modified Bargmann group, with U(1) replaced by $\mathbb{R}$ compared to the lecture definition of the Bargmann group.[4]

Use this to argue that the free Schrödinger equation (2) is symmetric under this action of the modified Bargmann group.

Hint: Write χ *in the form from the hint in part (d), and call the free constant* $-\alpha$. *For the combination of two transformations, you have to perform a long computation in order to show you obtain the Bargmann group multiplication rule. The last part follows almost trivially from the previous considerations.*

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[3] Since from parts (a) and (b) we know that the combined action (pushforward by $\varphi_{(R,k)}$) $\circ$ (Milne boost by $R^{-1}k$) leaves our τ, h, v invariant, this action on a really defines an action of the inhomogeneous Galilei group on Bargmann structures.

[4] This means we are considering the group $\widetilde{\text{Barg}} = \text{Gal} \ltimes_\rho (\mathbb{R}^{n+1} \times \mathbb{R})$ with

$$\rho_{(R,k)}(y^A, \alpha) = (y^t, R^a{}_b y^b + y^t k^a, \alpha + \tfrac{1}{2}|k|^2 y^t + k_a R^a{}_b y^b). \tag{5}$$

[E24] *Newtonian limits of matter actions*

In the lecture, we are going to show that starting with a Lorentzian spacetime (M, g), by expanding a Lorentzian orthonormal frame $(\mathrm{E}_0, \mathrm{E}_a)$ and its dual frame $(\mathrm{E}^0, \mathrm{E}^a) = (-(\mathrm{E}_0)^\flat, (\mathrm{E}_a)^\flat)$ as

$$\mathrm{E}_0 = c^{-1}v + \mathrm{O}(c^{-3}), \qquad \mathrm{E}_a = \mathrm{e}_a + \mathrm{O}(c^{-2}), \tag{6a}$$

$$\mathrm{E}^0 = c\tau + c^{-1}a + \mathrm{O}(c^{-3}), \qquad \mathrm{E}^a = \mathrm{e}^a + \mathrm{O}(c^{-2}), \tag{6b}$$

for a chosen τ, we obtain a Galilei frame (v, e_a) and the corresponding extended coframe $(\tau, \mathrm{e}^a, \mathrm{i}a)$ for a Galilei manifold with a Bargmann structure.

In this exercise, we are going to show how to obtain the point particle and Schrödinger field actions from the lecture as formal Newtonian limits of Lorentzian counterparts.

(a) We consider the action for the timelike future-directed worldline γ of a massive point particle action in Lorentzian spacetime,

$$S_{\mathrm{Lor.}}[\gamma] = -mc \int_{\lambda_i}^{\lambda_f} \mathrm{d}\lambda \sqrt{-g(\gamma', \gamma')}\,. \tag{7}$$

Show that subtracting the 'rest energy' term $S_{\mathrm{rest}}[\gamma] = -mc^2 \int_\gamma \tau$, in the limit $c \to \infty$ this action goes over to the action for a massive point particle in a Galilei manifold that was introduced in the lecture in construction 5.11.

Hint: Insert $g = \eta_{AB}\mathrm{E}^A \otimes \mathrm{E}^B = -\mathrm{E}^0 \otimes \mathrm{E}^0 + \delta_{ab}\mathrm{E}^a \otimes \mathrm{E}^b$, *use (6), and expand the action using the power series expansion of* $\sqrt{1+x}$.

(b) Consider the action for a Klein–Gordon field Φ on Lorentzian spacetime,

$$S_{\mathrm{KG}}[\Phi] = -c^{-1} \int \mathrm{vol}_g \left(\hbar^2 g^{\mu\nu}\, \overline{\partial_\mu \Phi}\, \partial_\nu \Psi + m^2c^2|\Phi|^2\right). \tag{8}$$

Assuming that $\tau = \mathrm{d}t$ is exact and separating off an oscillating 'rest energy' phase factor with positive frequency from the Klein–Gordon field according to

$$\Phi = \mathrm{e}^{-\mathrm{i}\frac{mc^2}{\hbar^2}t}\Psi, \tag{9}$$

show that in the limit $c \to \infty$ the Klein–Gordon action goes over to the Schrödinger field action in a Galilei spacetime for Ψ, as introduced in the lecture.

Hint: First show that $\partial_\mu \Phi = \mathrm{e}^{-\mathrm{i}\frac{mc^2}{\hbar^2}t}(-\mathrm{i}\frac{mc^2}{\hbar}\tau_\mu + \partial_\mu)\Psi$. *When then inserting the metric in terms of the orthonormal frame, note that you can compute the expression* $c\mathrm{E}_0^\mu \tau_\mu$ *to order* c^{-2} *by using the dual basis condition* $1 = \mathrm{E}_0^\mu \mathrm{E}_\mu^0$, *and similarly* $\mathrm{E}_a^\mu \tau_\mu$ *to* c^{-2} *using* $0 = \mathrm{E}_a^\mu \mathrm{E}_\mu^0$.