The Bargmann group, and the NUT region – again

[E20] *The Bargmann homomorphisms*

Let $\rho\colon \mathsf{Gal} \to \mathsf{Aut}(\mathbb{R}^{n+1} \times \mathsf{U}(1))$ be the homomorphism from the definition of the Bargmann group, given by

$$\rho_{(R,k)}(y^A, \mathrm{e}^{\mathrm{i}\varphi}) = \left(y^t, R^a{}_b y^b + y^t k^a, \mathrm{e}^{\mathrm{i}(\varphi + \frac{1}{2}|k|^2 y^t + k_a R^a{}_b y^b)}\right). \tag{1}$$

Compute the induced homomorphisms $\dot\rho\colon \mathsf{Gal} \to \mathsf{Aut}(\mathbb{R}^{n+1} \oplus \mathfrak{u}(1))$ and $\dot\rho'\colon \mathfrak{gal} \to \mathsf{Der}(\mathbb{R}^{n+1} \oplus \mathfrak{u}(1))$, defined as $\dot\rho_{(R,k)} := \mathrm{D}(\rho_{(R,k)})\Big|_{(0,1)}$ and $\dot\rho' := \mathrm{D}\dot\rho|_{(\mathbb{1},0)}$.

[E21] *The Newtonian limit of the NUT region via Bargmann structures*

In the lecture, we are going to show that starting with a Lorentzian spacetime, by expanding a Lorentzian orthonormal frame $(\mathrm{E}_0, \mathrm{E}_a)$ and its dual frame $(\mathrm{E}^0, \mathrm{E}^a) = (-(\mathrm{E}_0)^\flat, (\mathrm{E}_a)^\flat)$ as

$$\mathrm{E}_0 = c^{-1} v + \mathrm{O}(c^{-3}), \qquad \mathrm{E}_a = \mathrm{e}_a + \mathrm{O}(c^{-2}), \tag{2a}$$
$$\mathrm{E}^0 = c\tau + c^{-1} a + \mathrm{O}(c^{-3}), \qquad \mathrm{E}^a = \mathrm{e}^a + \mathrm{O}(c^{-2}), \tag{2b}$$

where we assume $\mathrm{d}\tau = 0$, we obtain a Galilei frame (v, e_a) and the corresponding extended coframe $(\tau, \mathrm{e}^a, \mathrm{i}a)$ for a Galilei manifold with absolute time with a Bargmann structure. We are furthermore going to show that the Galilei connection that arises as the formal $c \to \infty$ limit of the Lorentzian Levi–Civita connection (compare theorem 2.35 (iii)) has vanishing extended torsion with respect to the limiting Bargmann structure.

In this exercise, we are going to use this to compute the Newton–Cartan limit of the NUT region of Taub–NUT spacetime (compare exercise [E14] from last semester's course), and show that it doesn't have absolute rotation, in a very efficient way.

The Taub–NUT metric is

$$g = U(c\mathrm{d}t - 4l\sin^2(\tfrac{\theta}{2})\,\mathrm{d}\varphi)^2 - \frac{1}{U}\mathrm{d}r^2 + (r^2 + l^2)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2), \tag{3a}$$

where

$$U = -1 + \frac{2\left(\frac{GM}{c^2} r + l^2\right)}{r^2 + l^2}. \tag{3b}$$

The inverse metric is given by

$$g^{-1} = \left(\frac{1}{U} + \frac{4l^2 \tan^2(\frac{\theta}{2})}{r^2 + l^2}\right) c^{-2} \partial_t \otimes \partial_t + \frac{l / \cos^2(\frac{\theta}{2})}{(r^2 + l^2)} c^{-1} (\partial_t \otimes \partial_\varphi + \partial_\varphi \otimes \partial_t)$$
$$- U \, \partial_r \otimes \partial_r + \frac{1}{r^2 + l^2} (\partial_\theta \otimes \partial_\theta + \sin^{-2}\theta \, \partial_\varphi \otimes \partial_\varphi). \tag{3c}$$

We parametrise $l = \frac{J}{Mc}$, and consider the *NUT region* $U < 0$.

(a) From the metric (3a), read off an orthonormal dual frame $(\mathrm{E}^0, \mathrm{E}^a)$ of one-forms. Compute the frame $(\mathrm{E}_0, \mathrm{E}_a) = (-(\mathrm{E}^0)^\sharp, (\mathrm{E}^a)^\sharp)$, where a 'sharp' sign denotes the metric-induced isomorphism sending dual vectors to vectors, i.e. $\alpha^\sharp = g^{-1}(\alpha, \cdot)$.

Hint: Since the metric is written as a sum of squares, you can directly read off the dual frame.

(b) By expanding the frame and dual frame as formal power series in c^{-1} and comparing to (2), determine the induced Galilei frame and corresponding extended coframe for the limiting Galilei manifold with Bargmann structure.

Hint: Don't forget to insert $l = \frac{J}{mc}$. *For inverting formal power series, use the geometric series. The resulting spatial frames should simply be orthonormal (dual) frames for Euclidean* $\mathbb{R}^3$ *in spherical coordinates,* $\tau = \mathrm{d}t$, *and* $v = \partial_t$.

(c) Argue that the resulting v is rigid, i.e. that $\mathcal{L}_v h = 0$.

Hint: Using $h = \delta^{ab} \mathrm{e}_a \otimes \mathrm{e}_b$, *this is almost obvious, given the resulting* v *and* e_a.

(d) Using the fact that the extended torsion vanishes, and the relationship between the Newton–Coriolis form, the mass torsion, and the extended coframe (corollary 5.7), compute the Newton–Coriolis form of the limiting Galilei connection with respect to v. Comparing to the decomposition $\Omega = \tau \wedge \alpha + 2\omega$, read off the twist ω and observe that it is spatially non-constant, i.e. that we don't have absolute rotation.

Using the formalism of Bargmann structures and extended coframes, we can thus compute the Newton–Coriolis form of a limiting Galilei manifold with respect to a *chosen* reference vector field v far more easily than just using expansions of the metric (compare the amount of calculation required in this exercise to that from [E14]): we just have to find an Lorentzian orthonormal frame for which the expansion of the timelike vector field E_0 starts with the chosen v, and then expand. (For expansions of the metric, a natural reference vector field v comes out of the expansion, and one cannot specify this beforehand.)