Semidirect products and sums, and fundamental vector fields

[E17] *Semidirect products and sums really are groups and Lie algebras*
In this exercise, we verify that semidirect products of groups really are groups and that semidirect sums of Lie algebras are Lie algebras.

(a) Let H, N be groups and $\rho\colon H \to \mathsf{Aut}(N)$ a group homomorphism. Show that

   (i) the group operation on the semidirect product $H \ltimes_\rho N$ as defined in the lecture really is associative,

   (ii) $(\mathrm{e}_H, \mathrm{e}_N)$ is its neutral element (where e_H and e_N are the neutral elements of H and N, respectively), and

   (iii) inverses are given by $(h,n)^{-1} = (h^{-1}, \rho_{h^{-1}}(n^{-1}))$.

(b) Let $\mathfrak{h}, \mathfrak{n}$ be Lie algebras and $\hat{\rho}\colon \mathfrak{h} \to \mathsf{Der}(\mathfrak{n})$ a Lie algebra homomorphism. Show that the Lie bracket on the semidirect sum $\mathfrak{h} \oplus_{\hat{\rho}} \mathfrak{n}$ really satisfies the Jacobi identity.

Hint: There's nothing you can do but just compute, using the properties of ρ and $\hat{\rho}$.

[E18] *The Lie algebra of a semidirect product Lie group*
Let H, N be Lie groups and $\rho\colon H \to \mathsf{Aut}(N)$ a Lie group homomorphism. We know that *as a vector space*, the Lie algebra of $H \ltimes_\rho N$ is $T_{(\mathrm{e}_H,\mathrm{e}_N)}(H \ltimes_\rho N) = T_{\mathrm{e}_H}H \oplus T_{\mathrm{e}_N}N = \mathfrak{h} \oplus \mathfrak{n}$. From the lecture (proposition 4.4 (ii)), we know that the adjoint representation of $H \ltimes_\rho N$ is given by

$$\mathrm{Ad}_{(h,n)}(X,Y) = (\mathrm{Ad}_h(X), \mathrm{Ad}_n(\dot{\rho}_h(Y)) + \sigma_n(\mathrm{Ad}_h(X))) \qquad (1)$$

for $(h,n) \in H \ltimes_\rho N$ and $X \in \mathfrak{h}, Y \in \mathfrak{n}$, where $\dot{\rho}\colon H \to \mathsf{Aut}(\mathfrak{n})$ is the homomorphism induced by ρ, and $\sigma_n\colon \mathfrak{h} \to \mathfrak{n}$ is the differential of the map $H \to N, h \mapsto n\rho_h(n^{-1})$ at the neutral element e_H.

Using the fact that the Lie bracket on the Lie algebra $\mathrm{Lie}(H \ltimes_\rho N)$ of our Lie group may be expressed as $[(X,Y),(\tilde{X},\tilde{Y})] = \mathrm{ad}_{(X,Y)}(X,Y)$ with

$$\mathrm{ad} = \mathrm{D}(\mathrm{Ad})|_{(\mathrm{e}_H,\mathrm{e}_N)}, \qquad (2)$$

show that the Lie algebra is indeed the semidirect sum $\mathfrak{h} \oplus_{\dot{\rho}'} \mathfrak{n}$, as claimed in the lecture.

Hint: Compute Lie brackets of the form $[(X,0),(\tilde{X},0)], [(0,Y),(0,\tilde{Y})], [(X,0),(0,Y)]$ by differentiating Ad*, and use those to express a general bracket. For differentiating* Ad*, consider curves in $H \ltimes_\rho N$ through the neutral element with specified derivative there, and use the chain rule, similar to the computation of* Ad *in the lecture by differentiating the conjugation map.*

(Please turn over)

[E19] *Translation of fundamental vector fields*

Here we will prove a small fact about fundamental vector fields of Lie group actions that we will use in the lecture.

Let G be a Lie group, with a smooth *right* action on a manifold M that we denote by

$$G \times M \ni (g, p) \mapsto p \cdot g = \mathrm{R}_g(p) \in M. \tag{3}$$

For $X \in \mathfrak{g}$, the corresponding fundamental vector field on M is defined by

$$\tilde{X}(p) := \frac{\mathrm{d}}{\mathrm{d}t} p \cdot g(t) \bigg|_{t=0} \quad \text{for a curve } g(t) \text{ in } G \text{ with } \dot{g}(0) = X, \tag{4a}$$

for example

$$\tilde{X}(p) = \frac{\mathrm{d}}{\mathrm{d}t} p \cdot \exp(tX) \bigg|_{t=0}. \tag{4b}$$

Show that fundamental vector fields satisfy

$$(\mathrm{R}_{g^{-1}})_* \tilde{X} = \widetilde{\mathrm{Ad}_g(X)}. \tag{5}$$

What would the corresponding equation look like for *left* actions?

Hint: Explicitly write out the definition of the pushforward vector field evaluated at a point p, and use the definition of Ad_g as the differential of the conjugation map $\tilde{g} \mapsto g\tilde{g}g^{-1}$ at the neutral element.