The Newtonian limit of the NUT region

Taub–NUT spacetime is an exact solution of the vacuum Einstein equations with some very peculiar properties. The Taub–NUT metric can be written as

$$g = U(c\mathrm{d}t - 4l\sin^2(\tfrac{\theta}{2})\,\mathrm{d}\varphi)^2 - \frac{1}{U}\mathrm{d}r^2 + (r^2+l^2)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2), \tag{1a}$$

where

$$U = -1 + \frac{2\left(\frac{GM}{c^2}r + l^2\right)}{r^2+l^2}. \tag{1b}$$

Here M is a mass and $l > 0$ is the so-called *NUT parameter*, θ and φ are usual spherical coordinates on S^2, and (at least a priori[1]) $t, r \in \mathbb{R}$. The inverse metric can be calculated as

$$g^{-1} = \left(\frac{1}{U} + \frac{4l^2\tan^2(\frac{\theta}{2})}{r^2+l^2}\right)c^{-2}\partial_t\otimes\partial_t + \frac{l/\cos^2(\frac{\theta}{2})}{(r^2+l^2)}c^{-1}(\partial_t\otimes\partial_\varphi + \partial_\varphi\otimes\partial_t)$$
$$- U\,\partial_r\otimes\partial_r + \frac{1}{r^2+l^2}(\partial_\theta\otimes\partial_\theta + \sin^{-2}\theta\,\partial_\varphi\otimes\partial_\varphi). \tag{1c}$$

The *NUT region* is the region in which $U < 0$, i.e. the region in which t is a timelike and r a spacelike coordinate. It is the region $r \notin [r_-, r_+]$, where $r_\pm = \frac{GM}{c^2} \pm \sqrt{\frac{G^2M^2}{c^4} + l^2}$.

Parametrising $l = \frac{J}{Mc}$, where J has the dimension of angular momentum, the metric in the NUT region has the form which can be expanded in c^{-1} to give a Newton–Cartan spacetime. (Note that in the limit $c \to \infty$, we have $r_\pm \to 0$, such that we may use the whole positive real axis as values for r.) The objective of this exercise sheet is to compute the formal $c \to \infty$ Newton–Cartan limit of the NUT region and show that it does not have absolute rotation.

[E14] *The Newtonian limit of the NUT region*

(a) Inserting $l = \frac{J}{Mc}$, expand the Taub–NUT metric and inverse metric as formal power series in c^{-1}, to order c^0 for g and order c^{-2} for g^{-1}. Comparing to the expansions from lemma 2.33 from the lecture, read off the tensor fields τ, h, k and $g^{(0)}$.

Hint: Use the geometric series. You should obtain $\tau = \mathrm{d}t$, and h should just be the inverse Euclidean metric on $\mathbb{R}^3$ in spherical coordinates.

[1] To consistently interpret the Taub-NUT solution, one needs to make some topological identifications. The usual way is to define, in our notation, $\tilde{t} := r$ and $\psi := \frac{ct}{2l} - \varphi$, and then interpret ψ, θ, φ as Euler coordinates on S^3, s.t. ψ becomes 4π-periodic.

(b) Compute the unit timelike vector field $v = -k(\tau, \cdot)$. Show that it is not rigid.

Hint: $v = \partial_t - \frac{J/\cos^2(\frac{\theta}{2})}{Mr^2}\partial_\varphi$. *For non-rigidity, you can argue that* $\mathcal{L}_v h$ *is non-zero without completely computing it.*

(c) Compute $\phi = -\frac{1}{2}g^{(0)}(v, v)$, and then the covariant space metric with respect to v as $\underset{v}{h} = g^{(0)} + 2\phi\tau \otimes \tau$.

(d) By remark 2.36 from the lecture, the Newton–Coriolis form of the limiting Newtonian connection with respect to v is given by $\Omega = \tau \wedge \mathrm{d}\phi$. Compute this.

(e) We want to show that the limiting Newton–Cartan spacetime does not have absolute rotation, i.e. that rigid unit timelike vector fields on it have spatially non-constant twist. Our field v from above has no twist, but it is not rigid. We consider the rigid field $\tilde{v} = \partial_t$ instead. Using the transformation behaviour of the Newton–Coriolis form under a Milne boost (equation (1.35c) from the lecture notes), compute the Newton–Coriolis form $\tilde{\Omega}$ with respect to $\tilde{v}$. Comparing to the decomposition $\tilde{\Omega} = \tau \wedge \tilde{\alpha} + 2\tilde{\omega}$, read off the twist $\tilde{\omega}$ and observe that it is non-constant.