NEWTON–CARTAN COSMOLOGY

[E11] *The Raychaudhuri equation*

Let (M, τ, h) be a Galilei manifold with absolute time, and ∇ a torsion-free Galilei connection on it. Let v be a unit timelike vector field, and denote by $\alpha, \omega, \theta, \sigma$ its acceleration, twist, expansion and shear fields with respect to ∇, respectively. Show that the rate of change of the expansion along v is[1]

$$\nabla_v \theta = -\frac{1}{n}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} + \nabla_\mu \alpha^\mu - R_{\mu\nu}v^\mu v^\nu, \tag{1}$$

where $n = \dim M - 1$, and $R_{\mu\nu}$ are the components of the Ricci tensor of ∇.

Why does this equation (in some sense) say that 'gravity is attractive'?

Hint: Write $\theta = \nabla_\mu v^\mu$, apply the Ricci identity ${R^\mu}_{\nu\rho\sigma}X^\nu = 2\nabla_{[\rho}\nabla_{\sigma]}X^\mu$, and use the decomposition of ∇v in terms of the various fields (equation (2.4) from the lecture notes).

[E12] *FLRW-like Newton–Cartan cosmology*

By a *Newton–Cartan cosmological model*, we will mean a Galilei manifold (M, τ, h) with absolute time with a torsion-free Galilei connection ∇, together with a unit timelike vector field ξ, interpreted as the spacetime velocity[2] of the cosmological matter, and a function ρ, interpreted as the mass density of the cosmological matter, such that the Newton–Cartan field equation $R_{\mu\nu} = 4\pi G\rho\tau_\mu\tau_\nu$ is satisfied. Note that due to the field equation ρ is redundant, since it is determined by the other data of the model.

A *symmetry* of a cosmological model $(M, \tau, h, \nabla, \xi)$ is a diffeomorphism $\varphi\colon M \to M$ that leaves all of the data invariant, i.e. satisfies

$$\varphi^*\tau = \tau,\ \varphi^*h = h,\ \varphi^*\nabla = \nabla,\ \varphi^*\xi = \xi. \tag{2}$$

The model is *(spatially) homogeneous and isotropic* iff for any spatial rotation $R \in \mathrm{SO}(\ker\tau|_p, {}^{(n)}h)$ there exists a symmetry φ with $\varphi(p) = p$ and $(\mathrm{D}\varphi|_p)|_{\ker\tau|_p} = R$ ('all spatial rotations are realisable by symmetries').[3]

Let $(M, \tau, h, \nabla, \xi)$ be a homogeneous and isotropic Newton–Cartan cosmological model.

(Please turn over)

[1] The exact same equation is true in GR, and is there an important tool for example in the proof of the various singularity theorems.

[2] Since we don't restrict to the case of four spacetime dimensions, we won't call this the 'four-velocity'.

[3] In $n = 3$ spatial dimensions, the condition that each spatial rotation be realisable by a symmetry is equivalent to demanding that for any two unit spacelike vectors at a point there exist a symmetry mapping one onto the other ('no preferred spacelike directions'). However, in higher dimensions this equivalence fails: for $n > 4$, there are proper subgroups of $\mathrm{SO}(n)$ whose action on S^{n-1} is transitive.

(a) Show that any vector field determined by τ, h, ∇, ξ is proportional to ξ.

Hint: Such a vector field is invariant under all symmetries. It can be written as $a\xi + w$ *with* w *spacelike, and then isotropy implies . . .*

(b) Deduce that ξ is geodesic, and that ρ is spatially constant.

(c) Show that any spacelike symmetric contravariant degree-2 tensor field $\lambda = \lambda^{\mu\nu}\partial_\mu\partial_\nu$ determined by τ, h, ∇, ξ is proportional to h.

Hint: At any point p, λ *can be expressed as* $\lambda|_p = \sum_{a=1}^{n} b^a \, \mathrm{e}_a \otimes \mathrm{e}_a$, *where the* e_a *form an orthonormal basis of spacelike vectors. To this, apply isotropy.*

(d) Deduce that ξ has vanishing shear.

(e) Show that for $n > 2$, any purely spacelike 2-form λ determined by τ, h, ∇, ξ vanishes.

Hint: At any point p, λ *can be expressed as* $\lambda|_p = \sum_{a<b}^{n} \lambda_{ab} \, \mathrm{e}^a \wedge \mathrm{e}^b$, *where the* e^a *are dual to an orthonormal basis of spacelike vectors. Consider isotropy for rotations that swap two basis vectors* e_c *and* e_d.

(f) Deduce that ξ has vanishing twist.

(g) Why do the preceding statements show that ∇ is Newtonian?

Hint: Newton–Coriolis form!

(h) Assuming that the spacetime is spatially flat, show that it has absolute rotation.

Hint: Construct a rigid twist-free vector field v *from* ξ, *by 'subtracting the expansion'. This works similar to the construction of twist-free vector fields in proposition 2.28 from the lecture.*

We thus have shown that in any homogeneous and isotropic Newton–Cartan cosmological model, the spacetime velocity of the cosmological matter is geodesic, shear- and twist-free, and its mass density is spatially constant. Furthermore, we showed that spacetime is Newtonian (i.e. that we have a proper model of Newton–Cartan gravity), and that assuming spatial flatness we get absolute rotation.

[E13] *Newtonian cosmology – not inconsistent!*
Let $(M, \tau, h, \nabla, \xi)$ be a spatially flat homogeneous and isotropic cosmological Newton–Cartan model, and let ρ be the corresponding mass density function. Here we are going to show that the solutions of ordinary Newtonian gravity we get from this by applying the Trautman recovery theorem are precisely the canonical solutions to Newtonian cosmology we know from exercise [E9].

(a) Let v be *any* rigid twist-free vector field, and let ϕ be a corresponding Newtonian gravitational potential from the Trautman recovery theorem. Show that it satisfies

$$\nabla^\mu \nabla^\nu \phi = \frac{4}{n} \pi G \rho h^{\mu\nu}, \tag{3}$$

where $n = \dim M - 1$.

*Hint: Express the fact that ξ is geodesic (which we know from **[E12]** (b)) via the Trautman recovery theorem, and use this to compute $\nabla^\mu \nabla^\nu \phi = \overset{v}{\nabla}{}^\mu \overset{v}{\nabla}{}^\nu \phi$. Then express $\overset{v}{\nabla}_\xi \xi$ in terms of $\nabla_\xi \xi$, rewrite this in terms of the expansion of ξ, and use the Raychaudhuri equation (1).*

(b) Using exercise [E10] from the previous sheet, show that for *any* function $\tilde{\phi}$ satisfying (3), there is a rigid twist-free vector field $\tilde{v}$ such that $\tilde{\phi}$ is a Newtonian potential with respect to $\tilde{v}$ from the Trautman recovery theorem.

Hint: Consider the difference $\phi - \tilde{\phi}$ to construct the Milne boost relating v to $\tilde{v}$.

As we have shown in exercise [E9] (c), (3) is (for $\rho > 0$) equivalent to ϕ being a canonical solution to Poisson's equation (on the corresponding spatial leaf), so we have in fact recovered the canonical solutions of Newtonian cosmology, which are seemingly non-homogeneous and non-isotropic, from a perfectly homogeneous and isotropic cosmological model.