Non-absolute rotation, and Newtonian cosmology

[E8] *Non-absolute rotation*

In this exercise, we are going to construct a Newton–Cartan spacetime that fails to satisfy the *absolute rotation condition* we are going to see in the lecture, i.e. in which rigid vector fields have spatially non-constant twist.

In the lecture we will see that starting from a model of usual Newtonian gravity, including rotation of reference frames, we can construct a Newton–Cartan spacetime ('geometrisation', see remark 2.23). In adapted coordinates (construction 2.22), the field equations for the twist and acceleration fields read

$$\partial_{[a}\omega_{bc]} = 0, \quad \partial_a \omega^{ab} = 0, \tag{1a}$$

$$\partial_{[a}\alpha_{b]} = \partial_t \omega_{ab}, \quad \partial_a \alpha^a = 4\pi G \rho - \omega_{ab}\omega^{ab}. \tag{1b}$$

We want to find a solution to these equations where the ω^{ab} are spatially non-constant.

(a) Explain why we need to work in $n \geq 3$ spatial dimensions to find solutions with spatially varying ω.

(b) In $n = 3$ dimensions, we may write any spatial two-form as $\omega_{ab} = -\varepsilon_{abc}\omega^c$, where ε_{abc} is the usual three-dimensional totally antisymmetric symbol[1]. Show that then the field equations (1a) take the form

$$\vec{\nabla} \cdot \vec{\omega} = 0, \quad \vec{\nabla} \times \vec{\omega} = 0, \tag{2}$$

where we use the standard notation for divergence and curl in vector calculus on three-dimensional Euclidean space.

(c) Find a non-constant vector field $\vec{\omega}$ on $\mathbb{R}^3$ that satisfies (2).

Hint: You can simply guess a solution. Take $\omega^x(x, y, z)$ to be the easiest non-constant function you can think of, and work from there.

(d) Rewrite (1b) in terms of $\vec{\omega}$ and usual vector calculus operations. Insert your $\vec{\omega}$ from (c) and find a solution $\vec{\alpha}$ for the case $\rho = 0$.

(e*) *Bonus question:* Interpret the resulting Newton–Cartan spacetime physically, if that's possible.[2]

On the exercise sheet after next, we will see an example of a general-relativistic spacetime with a Newtonian limit that doesn't have absolute rotation.

[1] Geometrically, it may be understood as the components of the natural volume form on Euclidean space in orthonormal coordinates.

[2] I don't know if it is.

[E9] *Newtonian cosmology – inconsistent?*[3]

We consider standard Newtonian gravity in its usual coordinate formulation; i.e. we have a gravitational potential ϕ satisfying Poisson's equation

$$\Delta\phi = 4\pi G\rho, \tag{3}$$

where $\Delta = \sum_{a=1}^{3} \partial_a^2$ is the Laplacian on three-dimensional Euclidean space and ρ is the mass density. We want to analyse Newtonian *cosmology* for a homogeneous isotropic universe, i.e. we assume ρ to be constant.

(a) Show that the Newtonian gravitational field $g_a = -\partial_a\phi$ *cannot* be homogeneous.

Hint: If it were homogeneous, what would Poisson's equation imply?

(b) Show that, given any point $\vec{p} \in \mathbb{R}^3$, any function ϕ satisfying

$$\partial_a\phi(\vec{x}) = \frac{4}{3}\pi G\,(x_a - p_a)\rho \tag{4}$$

solves Poisson's equation. Such a solution we call a *canonical solution centred at* $\vec{p}$.

(c) Show that, if $\rho > 0$, ϕ being a canonical solution centred at some point (i.e. satisfying (4) for some $\vec{p}$) is equivalent to

$$\partial_a\partial_b\phi = \frac{4}{3}\pi G\rho\delta_{ab}. \tag{5}$$

Hint: Consider the difference of ϕ to a canonical solution $\tilde{\phi}$ centred at some $\tilde{\vec{p}}$.

(d) By explicitly writing down a canonical solution centred at $\vec{p}$, show that canonical solutions exist.

Hint: You should be able to guess a solution.

That the gravitational field in a homogeneous Newtonian universe cannot be homogeneous was in the past considered as evidence that Newtonian cosmology is inconsistent. The existence of several solutions centred at *any* point seems to make this even worse – which point should be the preferred one?

As it turns out, this apparent problem can be solved by realising that the 'Newtonian gravitational field' $-\partial_a\phi$ can never be measured on its own: one always measures the combination of gravitational and inertial forces, and one can show that this leads to canonical solutions centred at any two different points being empirically indistinguishable.

On the next exercise sheet, we will see how this problem can be solved in the context of Newton–Cartan gravity.

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[3] This exercise, and the corresponding one on the next sheet, are based on work by Malament.

[E10] *Newtonian potential under change of frame*

Let (M, τ, h, ∇) be a Newton–Cartan spacetime, and let v be a twist-free rigid unit timelike vector field. By the Künzle–Ehlers recovery theorem and twist-freeness, we know that the acceleration satisfies $\mathrm{d}(\alpha|_\Sigma) = 0$. Therefore, by the Poincaré lemma there is (locally) a function ϕ such that

$$\alpha|_\Sigma = \mathrm{d}(\phi|_\Sigma), \tag{6a}$$

or in components

$$\alpha^\mu = \nabla^\mu \phi. \tag{6b}$$

This ϕ is unique up to addition of spatially constant functions; and in the remaining field equation it plays the role of the Newtonian gravitational potential.

Let $\tilde{v}$ be another twist-free rigid field, and let k be the Milne boost field relating the two reference fields v and $\tilde{v}$, i.e. $k := v - \tilde{v}$. Show that a function $\tilde{\phi}$ is a potential for $\tilde{v}$ if and only if it is related to ϕ by

$$\nabla^\mu(\phi - \tilde{\phi}) = v^\rho \nabla_\rho k^\mu. \tag{7}$$

What does this equation look like in coordinates adapted to v?