**Exercise III** for Newton–Cartan gravity
Non-rotating directions
To be discussed in tutorial 3 on 17/11/2022

WiSe 2022/23
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## Non-rotating directions

**[E6]** *Non-rotating vectors in different coordinates*

We consider the $(2+1)$-dimensional Galilei manifold $M = \mathbb{R}^3$ with coordinates $(t, x^1, x^2) = (t, x, y)$, with clock form $\tau = \mathrm{d}t$ and space metric $h = \sum_{i=1}^{2} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^i}$. On this, we consider the torsion-free Galilei connection $\nabla$ whose connection coefficients in the coordinates $(t, x, y)$ vanish, $\Gamma^{\rho}_{\mu\nu} = 0$.

We also consider the unit timelike vector field $v = \frac{\partial}{\partial t} - \omega y \frac{\partial}{\partial x} + \omega x \frac{\partial}{\partial y}$, for some number $\omega$.

(a) Determine the flow line (/ integral curve) $\gamma_{(x_0,y_0)}(t)$ of $v$ determined by the initial condition $\gamma_{(x_0,y_0)}(0) = (0, x_0, y_0)$.

(b) In the lecture / exercise **[E7]**, we will see that in Newton–Cartan gravity, a vector field along the worldline of an observer is interpreted as defining a *non-rotating* direction from the observer's point of view iff it is parallely transported with respect to the Galilei connection of spacetime.

Show that the spacelike vector field $\frac{\partial}{\partial x}$ is non-rotating from the point of view of an observer moving along *any* of the worldlines $\gamma_{(x_0,y_0)}$.

*In coordinate components, the parallel transport equation $\nabla_{\dot\gamma} w = 0$ becomes $\dot w^\mu(t) + \Gamma^{\mu}_{\rho\sigma}(\gamma(t))\,\dot\gamma^\rho(t) w^\sigma(t) = 0$.*

(c) We now construct new coordinates $(\tilde t, \tilde x, \tilde y)$ that are 'adapted' to the observer vector field $v$, in the following way: for any point $p \in M$, there is a unique flow line of $v$ passing through it. We define $(\tilde x(p), \tilde y(p))$ to be the such that this flow line is $\gamma_{(\tilde x(p), \tilde y(p))}$. (This means that the new coordinate system can be imagined to be realised by coordinate labels that are mounted to particles flowing with $v$.) The time coordinate $\tilde t$ is the same as $t$. (In these new coordinates, the flow lines of $v$ are the lines of constant $\tilde x, \tilde y$, i.e. $v = \frac{\partial}{\partial \tilde t}$.

Determine the form of the non-rotating vector fields $\frac{\partial}{\partial x}$ in the new coordinates.

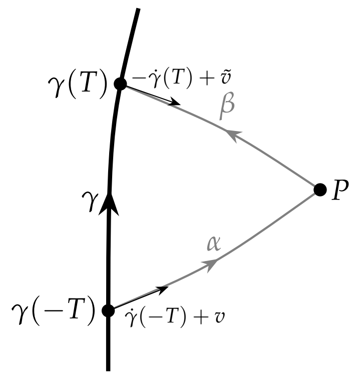
*Hint: Using your result from (a), express $(x, y)$ in terms of $(\tilde x, \tilde y)$ in matrix form. From this you can easily obtain $(\tilde x, \tilde y)$ in terms of of $(x, y)$, which allows you to compute $\frac{\partial}{\partial x}$ in terms of the new coordinates.*

**[E7]** *Operational definition of non-rotating directions*

For an observer moving on an arbitrary timelike worldline there is, a priori, no absolute notion of 'non-rotating' directions. This is the same in GR as in Newton–Cartan gravity. In this exercise, we will explore a physically sensible method to define operationally what is meant by 'the same direction' over the course of time, and see how this notion is realised mathematically.

The method works as follows[1]. Imagine an observer moving through spacetime. At some point in time, the observer throws away a ball with initial speed $|v|$ in some initial direction. The ball moves freely, and eventually gets 'reflected' back towards the observer in such a way that when it arrives again at the observer, its arrival speed is again $|v|$. The direction from which the ball arrives is then to be called 'the same direction' as the initial direction, at this later time.

More precisely, we work in a Galilei manifold $(M, \tau, h)$ with absolute time, with a torsion-free Galilei connection $\nabla$. We call the observer's worldline $\gamma$. At time $-T$, the ball is thrown with initial velocity $v \in \ker \tau|_{\gamma(-T)}$ (as seen from $\gamma$), and then moves along a freely falling worldline (i.e. geodesic of $\nabla$) $\alpha$. We want the ball to arrive again on $\gamma$ at time $T$ with velocity $-\tilde{v} \in \ker \tau|_{\gamma(T)}$, where $|v| = |\tilde{v}|$, moving on a geodesic $\beta$ that crosses $\alpha$ (in some event $P$, where the 'reflection' happened).




We will show below that this procedure determines $\tilde{v}$ to linear order in $T$, and that the map $v \mapsto \tilde{v}$ is given by parallel transport[2] along $\gamma$.

For simplicity, we parametrise all timelike curves by absolute time $t$, and work in coordinates $(t, x^i)$ where the $x^i$ are coordinates on the spatial leaves such that the observer's worldline $\gamma$ corresponds to $x^i = 0$ (i.e. we have $\gamma^t(t) = t, \gamma^i(t) = 0$). We also use the notation $N := \gamma(0)$.

(a) The first worldline $\alpha(t)$ of the ball has to satisfy the initial value problem

$$\nabla_{\dot{\alpha}} \dot{\alpha} = 0, \quad \alpha(-T) = \gamma(-T), \quad \dot{\alpha}(-T) = \dot{\gamma}(-T) + v. \tag{1}$$

By expressing this in the coordinates $(t, x^i)$ and Taylor-expanding $\alpha(t)$ around $t = -T$ to second order, determine $\alpha(t)$ approximately, for $t = \mathcal{O}(T)$.

*Hint: The spatial component of the geodesic equation is $\ddot{\alpha}^i(t) + \Gamma^i_{\mu\nu}(\alpha(t))\dot{\alpha}^\mu(t)\dot{\alpha}^\nu(t)$. From this, you can obtain $\ddot{\alpha}^i(-T)$. Finally, note that you can write $\Gamma^i_{\mu\nu}(\gamma(-T)) = \Gamma^i_{\mu\nu}(N) + \mathcal{O}(T)$.*

[1] This method for operationally defining non-rotation is inspired by a GR textbook by Synge. There, instead of a ball, light is used.

[2] In GR, the transport of non-rotating directions is not given by parallel transport with respect to the Levi–Civita connection, but instead by so-called *Fermi–Walker transport*.

(b) Similarly, determine the second worldline $\beta(t)$ of the ball, solving

$$\nabla_{\dot\beta}\dot\beta = 0, \quad \beta(T) = \gamma(T), \quad \dot\beta(T) = \dot\gamma(T) - \tilde v, \tag{2}$$

by Taylor-expanding around $t = T$ to second order.

*Hint: You can obtain the solution directly from that for $\alpha$ from part (a) by the replacements $-T \to T$, $v \to -\tilde v$.*

(c) For an intersection point $P$ to exist, we need the equation $\alpha(t_P) - \beta(t_P) = 0$ to have a solution $t_P$. By expanding $t_P = t_P^{(1)} T + t_P^{(2)} T^2 + \mathcal{O}(T^3)$ as a power series in $T$ and similarly for $\tilde v^i$, and inserting your approximate worldlines $\alpha(t), \beta(t)$, show that a solution exists iff we have

$$\tilde v^i = A v^i - 2T\,\Gamma^i_{tj}(N) v^j + \mathcal{O}(T^2) \tag{3}$$

for some number $A$.

*Hint: Before inserting the expansions into the equation, determine the leading-order coefficient of $\tilde v^i$ by thinking about what $\tilde v^i$ looks like for $T = 0$.*

(d) Show that parallel transport along $\gamma$ for time $2T$ maps $v$ to a vector with spatial components

$$v^i - 2T\,\Gamma^i_{tj}(N) v^j + \mathcal{O}(T^2). \tag{4}$$

*Hint: Solve the parallel transport equation*

$$\nabla_{\dot\gamma} V(t) = 0, \quad V(-T) = v \tag{5}$$

*via Taylor expansion around $t = -T$.*

Now comparing (3) and (4), we are finished: parallel transport of spacelike vectors conserves their length (due to $\nabla h = 0$); thus for $\tilde v$ to have the same length as $v$, we need $A = 1$ in (3) and $\tilde v$ is given by parallel transport.