The Newton–Coriolis form, and Milne boosts

[E4] *The Newton–Coriolis form of a general Galilei connection*

Consider a Galilei manifold (M, τ, h). Let v be a unit timelike reference vector field on it, and denote by P^μ_ν its associated projector and by $h_{\mu\nu}$ its associated covariant space metric. Let T be any tensor field satisfying

$$T^\rho{}_{\mu\nu} = -T^\rho{}_{\nu\mu} \ , \quad \tau_\rho T^\rho{}_{\mu\nu} = (\mathrm{d}\tau)_{\mu\nu} \ , \tag{1}$$

and Ω an arbitrary two-form.

In this exercise, we want to prove the remaining part of theorem 1.16 (b), namely that the Galilei connection defined by

$$\Gamma^\rho_{\mu\nu} = v^\rho \partial_{(\mu}\tau_{\nu)} + \frac{1}{2}h^{\rho\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) + \frac{1}{2}T^\rho{}_{\mu\nu} - T_{(\mu\nu)}{}^\rho + \tau_{(\mu}\Omega_{\nu)}{}^\rho \tag{2}$$

has Newton–Coriolis form Ω with respect to v. (*That* this defines a Galilei connection we know from the first part of theorem 1.16 (b) / exercise [E3]).

(a) First consider the Galilei connection $\overset{v}{\nabla}$ for $T^\rho{}_{\mu\nu} = v^\rho(\mathrm{d}\tau)_{\mu\nu}$, $\Omega = 0$, and show that it has vanishing Newton–Coriolis form wrt. v, i.e. that $2h_{\mu[\nu}\overset{v}{\nabla}_{\rho]}v^\mu = 0$.

Hint: You need to apply the product rule 'backwards' several times, and use the definitions of P^μ_ν and $h_{\mu\nu}$.

(b) Now writing the connection for general T and Ω as $\Gamma^\rho_{\mu\nu} = \overset{v}{\Gamma}{}^\rho_{\mu\nu} + S^\rho{}_{\mu\nu}$, the Newton–Coriolis form of ∇ has components

$$2h_{\mu[\nu}\nabla_{\rho]}v^\mu = 2\underbrace{h_{\mu[\nu}\overset{v}{\nabla}_{\rho]}v^\mu}_{\overset{(a)}{=}0} + 2h_{\mu[\nu}S^\mu{}_{\rho]\kappa}v^\kappa = 2S_{[\nu\rho]\kappa}v^\kappa. \tag{3}$$

Insert the explicit form of S and show that you obtain $\Omega_{\rho\nu}$.

(Please turn over)

[E5] *Milne boosts*

Let v be a unit timelike reference vector field on a Galilei manifold (M, τ, h), and let k be any spacelike vector field. The change

$$v^\mu \to \tilde{v}^\mu = v^\mu - k^\mu \tag{4}$$

of reference vector field is called the *Milne boost with respect to k*. Of course, if we change v, all the objects defined with respect to v change as well – here, we want to understand *how* they change.

(a) Show that under the Milne boost (4), the projector P with respect to v changes according to

$$P^\mu_\nu \to P^\mu_\nu + k^\mu \tau_\nu \tag{5}$$

(i.e. the right-hand side of this formula are the components of the projector with respect to $\tilde{v}$).

(b) Show that under the Milne boost (4), the covariant space metric $\underset{v}{h}$ with respect to v changes according to

$$h_{\mu\nu} \to h_{\mu\nu} + k_\mu \tau_\nu + \tau_\mu k_\nu + k^2 \tau_\mu \tau_\nu \tag{6}$$

(i.e. the right-hand side of this formula are the components of $\underset{\tilde{v}}{h}$).

Hint: Show by direct computation that the right-hand side satisfies the two defining equations for $\underset{\tilde{v}}{h}$. Be careful: you have to use the projector wrt. $\tilde{v}$!

(c) Let ∇ be a Galilei connection on (M, τ, h). Show that under the Milne boost (4), the Newton–Coriolis form Ω of ∇ with respect to v changes according to

$$\Omega_{\mu\nu} \to \Omega_{\mu\nu} - 2\partial_{[\mu} k_{\nu]} - (\partial_{[\mu} k^2) \tau_{\nu]} + k_\rho T^\rho{}_{\mu\nu} \ , \tag{7}$$

where the index on k_μ has been lowered with $\underset{v}{h}$, and $k^2 = {}^{(n)}h(k,k)$ is the squared length of k (i.e. show that if Ω is the Newton–Coriolis form of ∇ with respect to v, then the right-hand side of (7) are the components of the Newton–Coriolis form of ∇ with respect to $\tilde{v}$).

Hint: You have to be careful: in the definition of the Newton–Coriolis form, both the reference vector field v and the covariant space metric used to lower the index before antisymmetrisation change under the Milne boost. By inserting these and computing step by step, you should arrive at the desired result. You will need to apply the product rule 'backwards' once, and to use lemma 1.15 to express $\nabla \underset{v}{h}$ in terms of ∇v. You will also encounter a term of the form $2k_\lambda \nabla_\mu k^\lambda$, which you can without computation rewrite as $\nabla_\mu k^2$ using that the connection induced on the spacelike distribution is metric wrt. the induced metric ${}^{(n)}h$ (construction 1.7 from the lecture).