Galilei manifolds, the Frobenius theorem, Galilei connections

[E0] *An example Galilei manifold*
Convince yourself that $M = \mathbb{R}^{n+1} = \{(t, x^1, \ldots, x^n)\}$ with the tensor fields $\tau = \mathrm{d}t$, $h = \delta^{ab}\partial_a \otimes \partial_b$ (where $\partial_a = \frac{\partial}{\partial x^a}$ and a, b are summed from 1 to n) is a Galilei manifold. Does it have absolute time?

[E1] *The Frobenius theorem for kernel distributions*
The usual formulation of the Frobenius theorem is as follows: a distribution (i.e. subbundle) $B \subset TM$ is integrable iff it is involutive, i.e. iff for two B-valued vector fields, their commutator is again B-valued, i.e.

$$X, Y \in \Gamma(B) \implies [X, Y] \in \Gamma(B). \tag{1}$$

In this exercise, we show how this implies the form of the theorem that was used in the lecture, namely that for a one-form τ, the distribution $\ker \tau$ is integrable iff $\tau \wedge \mathrm{d}\tau = 0$.

(a) Let M be a manifold and $\tau \in \Omega^1(M)$ a one-form. Show that involutivity of the distribution $\ker \tau$ is equivalent to

$$v, w \in \ker \tau|_p \implies \mathrm{d}\tau|_p(v, w) = 0 \text{ for all } p \in M. \tag{2}$$

Hint: use the coordinate-free formula for the exterior derivative.

(b) Let V be a finite-dimensional vector space, $\alpha \in V^*$ a one-form on V and $\beta \in \bigwedge^2 V^*$ a two-form. Show that

$$v, w \in \ker \alpha \implies \beta(v, w) = 0 \tag{3}$$

is equivalent to $\alpha \wedge \beta = 0$.

Hint: First observe that (3) is equivalent to

$$\ker \alpha \subset \ker(\beta(v, \cdot)) \textit{ for all } v \in \ker \alpha. \tag{4}$$

Argue that this is equivalent to $\beta(v, \cdot) \propto \alpha$ for all $v \in \ker \alpha$ (think of the kernels' dimensions!). By extending α to a basis of V^* and expanding β in the induced basis of two-forms, you can show that this is equivalent to $\beta = \alpha \wedge \gamma$ for some one-form $\gamma \in V^*$. Finally, argue that this is equivalent to $\alpha \wedge \beta = 0$.

For those of you who have never worked with index notation for connections, here a small 'crash course'. Given a connection ∇, from any tensor field A we obtain a tensor field ∇A (the covariant derivative of A with respect to ∇) of one higher covariant degree than A (since we need to specify the direction in which we want to differentiate). If A has components $A^{\mu_1\dots\mu_s}_{\nu_1\dots\nu_t}$, then the components of ∇A are denoted by $(\nabla A)_\rho{}^{\mu_1\dots\mu_s}{}_{\nu_1\dots\nu_t} =: \nabla_\rho A^{\mu_1\dots\mu_s}_{\nu_1\dots\nu_t}$ and may (in a coordinate basis) be computed as

$$\begin{aligned}\nabla_\rho A^{\mu_1\dots\mu_s}_{\nu_1\dots\nu_t} &= \partial_\rho A^{\mu_1\dots\mu_s}_{\nu_1\dots\nu_t} + \Gamma^{\mu_1}_{\rho\lambda} A^{\lambda\mu_2\dots\mu_s}_{\nu_1\dots\nu_t} + \dots + \Gamma^{\mu_s}_{\rho\lambda} A^{\mu_1\dots\mu_{s-1}\lambda}_{\nu_1\dots\nu_t} \\ &\quad - \Gamma^{\lambda}_{\rho\nu_1} A^{\mu_1\dots\mu_s}_{\lambda\nu_2\dots\nu_t} - \dots - \Gamma^{\lambda}_{\rho\nu_t} A^{\mu_1\dots\mu_s}_{\nu_1\dots\nu_{t-1}\lambda}\end{aligned} \tag{5}$$

in terms of the components of A and the (coordinate) connection coefficients of ∇. The Leibniz / product rule for ∇ then takes the form

$$\nabla_\rho (A^{\mu\dots}_{\nu\dots} B^{\kappa\dots}_{\lambda\dots}) = (\nabla_\rho A^{\mu\dots}_{\nu\dots}) B^{\kappa\dots}_{\lambda\dots} + A^{\mu\dots}_{\nu\dots} \nabla_\rho B^{\kappa\dots}_{\lambda\dots} . \tag{6}$$

Note that this form of the product rule in index notation is not exactly the same as that in invariant/index-free notation that is used to *define* the action of ∇ on arbitrary tensor fields; however, they can be shown to be equivalent (which is not as obvious as it might seem!). In the following, you may just use this form of the product rule.

[E2] *Properties of Galilei connections*
Let ∇ be a Galilei connection on a Galilei manifold (M, τ, h).

(a) Show that the temporal torsion of ∇ is $d\tau$ (Proposition 1.6. (a) from the lecture), but use index notation.

Hint: Use that $(d\tau)_{\mu\nu} = 2\partial_{[\mu}\tau_{\nu]}$ *and* $T^\rho{}_{\mu\nu} = 2\Gamma^\rho_{[\mu\nu]}$*, where square brackets around indices denote total antisymmetrisation, e.g.* $X_{[\mu\nu]} = \frac{1}{2}(X_{\mu\nu} - X_{\nu\mu})$*, and that* $\nabla\tau = 0$*.*

(b) Show that the curvature tensor of ∇ satisfies $R^{\mu\nu}{}_{\rho\sigma} = -R^{\nu\mu}{}_{\rho\sigma}$, where the second index has been raised with h (second part of Proposition 1.6. (b)).

Hint: In index notation, the curvature tensor of any connection satisfies the identity

$$R^\mu{}_{\nu\rho\sigma} X^\nu = 2\nabla_{[\rho}\nabla_{\sigma]} X^\mu + T^\nu{}_{\rho\sigma} \nabla_\nu X^\mu \tag{7}$$

for any vector field X *('Ricci identity'). Use this to evaluate the expression* $2\nabla_{[\rho}\nabla_{\sigma]}(X^\mu Y^\nu)$ *for two arbitrary vector fields* X, Y*. Applying your result to* $h^{\mu\nu}$ *(which can be written as a sum of tensor products of two vector fields) and using* $\nabla h = 0$*, you should obtain the result.*

[E3] *Existence of Galilei connections*

Consider a Galilei manifold (M, τ, h).

(a) Let ∇ be a Galilei connection on (M, τ, h), and $\tilde{\nabla}$ an arbitrary connection. Denote their difference by $S^{\rho}{}_{\mu\nu} := \tilde{\Gamma}^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\mu\nu}$. Show that $\tilde{\nabla}$ is a Galilei connection iff S satisfies

$$S^{(\rho}{}_{\mu}{}^{\nu)} = 0, \tag{8a}$$

$$\tau_{\rho} S^{\rho}{}_{\mu\nu} = 0. \tag{8b}$$

(b) Let v be a unit timelike vector field on (M, τ, h) (i.e. a vector field with $\tau(v) = 1$), $P^{\mu}_{\nu} = \delta^{\mu}_{\nu} - v^{\mu}\tau_{\nu}$ its associated projection operator, and let $h_{\mu\nu} = h_{\nu\mu}$ be defined by

$$h_{\mu\nu} v^{\nu} = 0, \; h^{\mu\nu} h_{\nu\rho} = P^{\mu}_{\rho} \; . \tag{9}$$

Show that

$$\overset{v}{\Gamma}{}^{\rho}_{\mu\nu} = v^{\rho} \partial_{\mu} \tau_{\nu} + \frac{1}{2} h^{\rho\sigma} (\partial_{\mu} h_{\nu\sigma} + \partial_{\nu} h_{\mu\sigma} - \partial_{\sigma} h_{\mu\nu}) \tag{10}$$

defines a Galilei connection on (M, τ, h).

Hint: Unfortunately, you have to ('just') do the calculation. Showing $\overset{v}{\nabla}_{\mu}\tau_{\nu} = 0$ is easy; for $\overset{v}{\nabla}_{\mu} h^{\rho\sigma} = 0$ you need to apply the product rule 'backwards' a few times, and use the definitions of P^{μ}_{ν} and $h_{\mu\nu}$.

(c) Using the notation from part (b), show that

$$\Gamma^{\rho}_{\mu\nu} = v^{\rho} \partial_{(\mu} \tau_{\nu)} + \frac{1}{2} h^{\rho\sigma} (\partial_{\mu} h_{\nu\sigma} + \partial_{\nu} h_{\mu\sigma} - \partial_{\sigma} h_{\mu\nu}) + \frac{1}{2} T^{\rho}{}_{\mu\nu} - h_{\lambda(\mu} T^{\lambda}{}_{\nu)}{}^{\rho} + \tau_{(\mu} \Omega_{\nu)}{}^{\rho} \tag{11}$$

defines a Galilei connection on (M, τ, h), where T is any tensor satisfying

$$T^{\rho}{}_{\mu\nu} = -T^{\rho}{}_{\nu\mu} \; , \quad \tau_{\rho} T^{\rho}{}_{\mu\nu} = (\mathrm{d}\tau)_{\mu\nu} \; , \tag{12}$$

and Ω is an arbitrary two-form.

Hint: Consider the difference to the connection $\overset{v}{\nabla}$ from part (b), and use part (a).