Newton–Cartan gravity

Lecture notes academic years 2021/22, 2022/23

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Preface / Introduction

Lecture description

Newton–Cartan gravity is a reformulation of Newtonian gravity in geometric language, bringing it closer to General Relativity (GR) than the standard formulation. This allows for a coordinate-free understanding of how GR reduces to Newtonian gravity in the 'non-relativistic' limit. In this lecture course, we will explore Newton–Cartan gravity in detail and have a look at some modern developments.

Prerequisites: To follow the course, a good understanding of basic differential geometry (manifolds, differential forms, tensor fields, connections) is indispensable, so I suggest that participants should have taken (at least) an introductory course either in General Relativity or in Riemannian geometry. For the second part, an understanding of principal bundles and associated vector bundles is necessary; this can, however, also be (briefly) covered in additional sessions, according to demand.

Some history, plan of the lecture, and literature

The geometrised formulation of Newtonian gravity that is today known as Newton–Cartan gravity was developed by Élie Cartan (who also introduced, for example, the concept of differential forms) in 1923 [Car23; Car24] with the aim of clarifying the connection of GR to Newtonian gravity. In 1926 (published in 1928), Kurt Friedrichs rediscovered the formalism independently [Fri28]. Important further contributions to the theory were made by Andrzej Trautman in the 1960s [Tra63; Tra65], who modernised some of the notation and realised the importance of an additional geometric condition, and by Hans Peter Künzle in the 1970s [Kün72; Kün76], who gave the first complete presentation of the formalism in its modern form (for more details on the history in between, see the references of the articles cited here). In a 1981 article [Ehl81], Jürgen Ehlers discussed 'frame theory', a framework comprising both GR and Newton–Cartan gravity, and made precise some of the earlier results of Künzle on the relation of GR and Newtonian / Newton–Cartan gravity.

In the first part of this lecture course, we will develop and explore this 'classical'

Newton–Cartan theory as discussed by Künzle and Ehlers. The philosopher of physics David Malament has written a textbook [Mal12] on foundational issues of GR with an extensive chapter on Newton–Cartan gravity. This is, in my (the lecturer's) opinion, an excellent presentation of the subject, apart from one rather unfortunate point: Malament's textbook is aimed at students with no prior exposure to GR or differential geometry, and therefore proceeds often in a very elementary way. This is of course a great approach for an elementary textbook, but in my opinion it precludes appreciation of some geometric aspects of the theory. Therefore, in this course we will make use of differential-geometric methods more heavily. Nevertheless, Malament's book is the go-to resource if you want another perspective on results covered in the first half of this course.

The second part of the course will deal with a more modern, somewhat gauge-theoretic perspective on Newton–Cartan theory that started to develop in the 1980s in the context of the description of matter coupling to gravitational fields [DK84; Duv+85]. In 2011, this approach was revived / rediscovered by quantum field theorists / string theorists [And+11], and in 2015 this was put in a more explicitly differential-geometric formalism [GPR15], which generalised the perspective from the 1980s and related it to some other modern developments that had happened in the early 2010s (see below). It is this formalism developed in 2015, put in an even more explicitly geometric form, that we will cover in the second half of the course. For this, we will need the notions of principal fibre bundles and associated vector bundles, which *are not going to be introduced in the lecture*. If you have no experience with principal bundles, you need to get acquainted with the theory in order to follow the second half of the lectures (it is foundational for modern differential geometry, so there are lots of textbooks and lecture notes covering it). Probably, we will cover a little bit of principal bundle theory in some extra sessions prior to the corresponding lectures.

Newton–Cartan theory and certain generalisations thereof became a hot topic of research again in the early 2010s. On the one hand, this is due to the discovery that it can be employed in the effective description of effects in Galilei-relativistic condensed matter physics, starting with the quantum Hall effect [Son13], which led to a broad interest in the condensed matter community. On the other hand, 'stringy' versions of Newton–Cartan gravity have been constructed [And+12], which play a role in so-called 'non-relativistic', i.e. Galilei-relativistic, versions of string theory and in 'non-relativistic holography', i.e. Galilei-relativistic realisations of the 'holographic principle' that is believed to be an important aspect of string theory. However, since I (the lecturer) have no experience with either condensed matter theory or string theory, these aspects will not be covered in the lecture course. Nevertheless, these developments have also led to implications for the purely classical-gravitational case, on which we will hopefully get at least a small outlook at the end of the course.

Contents

1. Galilei manifolds

Here, we will introduce and develop the basic framework for the description of spacetime in Newton–Cartan gravity: so-called Galilei manifolds and Galilei connections on them.

1.1. Foundations

Definition 1.1. A *Galilei manifold* is an $(n+1)$-dimensional differentiable[1] manifold[2] M together with

(i) a nowhere-vanishing one-form $\tau \in \Omega^1(M)$, called the *clock form*, and

(ii) a symmetric contravariant degree-2 tensor field $h = h^{\mu\nu}\,\partial_\mu \otimes \partial_\nu \in \Gamma(TM \otimes TM)$, $h^{\mu\nu} = h^{\nu\mu}$, of signature $(n,0,1)$ (i.e. in a Sylvester basis, its components are $\operatorname{diag}(0,\underbrace{1,\ldots,1}_{n})$), called the *space metric*, such that

(iii) τ spans the degenerate direction of h, i.e.

$$\tau_\mu h^{\mu\nu} = 0. \tag{1.1}$$

A vector $v \in TM$ is *spacelike* iff $\tau(v) = 0$, and *timelike* otherwise. It is *future-directed* iff $\tau(v) > 0$.

The Galilei manifold has *absolute time* iff the clock form is closed, i.e. $\mathrm{d}\tau = 0$.

Points of M are interpreted as spacetime events. τ is interpreted as measuring time, i.e. the integral

$$\int_\gamma \tau = \int_\gamma \tau_\mu \mathrm{d}x^\mu = \int_{\lambda_\mathrm{i}}^{\lambda_\mathrm{f}} \mathrm{d}\lambda\, \frac{\mathrm{d}x^\mu(\lambda)}{\mathrm{d}\lambda}\, \tau_\mu(x(\lambda)) \tag{1.2}$$

of τ along a curve γ is the time elapsed along γ.

[1] For simplicity, we will always assume all objects to be smooth, i.e. C^∞. However, in almost all cases, a finite degree of differentiability is sufficient.

[2] We use the standard conventions of modern differential topology, i.e. manifolds are Hausdorff and second countable / paracompact.

If we assume $\tau \wedge \mathrm{d}\tau = 0$, then by the Frobenius theorem the distribution $\ker \tau \subset TM$ of spacelike vectors is integrable, i.e. M is foliated by submanifolds whose tangent spaces at each point are the respective spaces of spacelike vectors. Put differently, $\tau \wedge \mathrm{d}\tau = 0$ implies that spacetime is foliated by n-dimensional 'leaves of space'.

Assuming the even stronger condition $\mathrm{d}\tau = 0$, the time between two events is independent of the curve connecting them (under some mild topological assumptions): taking two curves γ_1, γ_2 connecting the same two points, by Stokes' theorem we have

$$\int_{\gamma_1} \tau - \int_{\gamma_2} \tau = \int_{\partial A} \tau = \int_A \mathrm{d}\tau = 0, \tag{1.3}$$

where A is any surface filling up[3] the boundary curve $\partial A = \gamma_1 \cup \overline{\gamma_2}$ (with the bar over γ_2 denoting reversed orientation). In this sense, a closed clock form defines an absolute notion of time, which is why we say that a Galilei manifold with closed clock form has absolute time. By the Poincaré lemma, this implies that locally $\tau = \mathrm{d}t$ for a function t, such that the spatial leaves from before are hypersurfaces of constant t, and the time between two events is the t difference between them.

Construction 1.2. The space metric h defines a positive-definite metric on the distribution of spacelike vectors (i.e. a positive-definite scalar product on spacelike vectors at any point $p \in M$) in the following way. $h|_p \in T_pM \otimes T_pM$ can be interpreted as a linear map $h|_p \colon T_p^*M \to T_pM$ (acting as $\alpha \mapsto h^{\mu\nu}|_p\, \alpha_\nu\, \partial_\nu|_p$). By definition this map has a one-dimensional kernel spanned by $\tau|_p$, and therefore induces an isomorphism

$$\widetilde{h}|_p \colon \; T_p^*M \Big/ \operatorname{span}\{\tau|_p\} \to \operatorname{im}(h|_p). \tag{1.4}$$

On the other hand, we know that the image of $h|_p$ consists of spacelike vectors, and for dimensional reasons it is the whole space of spacelike vectors, i.e. $\operatorname{im}(h|_p) = \ker \tau|_p$. Thus we can identify the space of spacelike vectors with the quotient space on the left-hand side of (1.4), on which $h|_p$ induces a positive-definite scalar product.

Concretely this means that for any two spacelike vectors $v, w \in \ker \tau|_p \subset T_pM$, there are covectors $\alpha, \beta \in T_p^*M$ with $v^\mu = h^{\mu\nu}\alpha_\nu$, $w^\mu = h^{\mu\nu}\beta_\nu$, and the scalar product between v and w is

$$\langle v, w \rangle = h(\alpha, \beta) = \alpha_\mu w^\mu = \beta_\mu v^\mu \; . \tag{1.5}$$

Here α and β are non-unique up to an addition of multiples of $\tau|_p$, which does not affect the resulting value.

Note that if the spacelike distribution is integrable (i.e. $\tau \wedge \mathrm{d}\tau = 0$), this means that h induces a Riemannian metric on each spatial leaf.

[3] The existence of this filling surface is where we need a topological assumption implying that the two curves are homotopic.

Definition 1.3. For an $(n+1)$-dimensional Galilei manifold (M, τ, h), we will sometimes denote the n-dimensional bundle metric induced by h on the spacelike distribution $\ker \tau$ by ${}^{(n)}h$.

Notation 1.4. When employing index notation for tensor fields on a Galilei manifold, we will raise indices using h. For example, if we have some tensor field with components $X^{\mu}{}_{\nu\rho}{}^{\sigma}{}_{\kappa}$, we use the notation

$$X^{\mu\nu}{}_{\rho}{}^{\sigma}{}_{\kappa} := h^{\nu\lambda} X^{\mu}{}_{\lambda\rho}{}^{\sigma}{}_{\kappa} \,. \tag{1.6}$$

Note that since h is degenerate, the operation of raising an index is not invertible, i.e. differently to the case in (pseudo-)Riemannian geometry we lose information when doing so.

1.2. Galilei connections

Similarly to the probably familiar case of Riemannian manifolds, we consider connections (i.e. covariant derivative operators) compatible with the structure of a Galilei manifold.

1.2.1. Definition and general properties

Definition 1.5. A *Galilei connection* on a Galilei manifold (M, τ, h) is a linear connection ∇ (i.e. a covariant derivative operator on the tangent bundle) satisfying

$$\nabla \tau = 0, \ \nabla h = 0. \tag{1.7}$$

The torsion of the connection will be denoted by T and the curvature tensor by R, i.e.

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y], \tag{1.8a}$$
$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{1.8b}$$

for vector fields X, Y, Z. In components, we use the usual index conventions, namely

$$T(X,Y) = T^{\rho}{}_{\mu\nu} X^{\mu} Y^{\nu} \partial_{\rho} \,, \tag{1.9a}$$
$$R(X,Y)Z = R^{\mu}{}_{\nu\rho\sigma} X^{\rho} Y^{\sigma} Z^{\nu} \partial_{\mu} \,. \tag{1.9b}$$

Proposition 1.6. *Let ∇ be a Galilei connection on (M, τ, h).*

(i) The temporal torsion of ∇ is $\mathrm{d}\tau$, i.e. we have $\tau(T(X,Y)) = \mathrm{d}\tau(X,Y)$ for any vectors (or vector fields) X, Y.

(ii) The curvature tensor of ∇ satisfies $\tau_\mu R^\mu{}_{\nu\rho\sigma} = 0$ and $R^{\mu\nu}{}_{\rho\sigma} = -R^{\nu\mu}{}_{\rho\sigma}$.

Proof. (i) Using compatibility of ∇ with τ, for any vector fields X, Y we obtain

$$\begin{aligned}\tau(T(X,Y)) &= \tau(\nabla_X Y) - \tau(\nabla_Y X) - \tau([X,Y]) \\ &= \nabla_X(\tau(Y)) - \nabla_Y(\tau(X)) - \tau([X,Y]) \\ &= X(\tau(Y)) - Y(\tau(X)) - \tau([X,Y]) \\ &= \mathrm{d}\tau(X,Y).\end{aligned} \tag{1.10}$$

(ii) Again using compatibility of ∇ with τ, for any vector fields X, Y, Z we have

$$\begin{aligned}\tau(R(X,Y)Z) &= \tau(\nabla_X\nabla_Y Z) - \tau(\nabla_Y\nabla_Y Z) - \tau(\nabla_{[X,Y]}Z) \\ &= X(\tau(\nabla_Y Z)) - Y(\tau(\nabla_X Z)) - [X,Y](\tau(Z)) \\ &= X(Y(\tau(Z))) - Y(X(\tau(Z))) - [X,Y](\tau(Z)) \\ &= 0.\end{aligned} \tag{1.11}$$

Expressed in index notation, this gives the first identity.

The compatibility of ∇ with h leads to the antisymmetry of the curvature tensor in the first two indices when raised, the details of which we leave as an exercise. □

In particular, we see that if $\mathrm{d}\tau \neq 0$, any Galilei connection necessarily has torsion. Put differently, torsion-free Galilei connections can exist only on Galilei manifolds with absolute time.

Construction 1.7. We consider the spacelike distribution $\ker\tau$. We claim that due to $\nabla\tau = 0$, any Galilei connection ∇ induces a natural connection $\overset{(n)}{\nabla}$ on $\ker\tau$ by restriction.

Indeed, given any vector $v \in TM$ and spacelike vector field X, the covariant derivative $\nabla_v X$ is again spacelike: we have $\tau|_p(\nabla_v X) = v(\tau(X)) = 0$ since X is spacelike.

Additionally, due to $\nabla h = 0$, the induced connection $\overset{(n)}{\nabla}$ is compatible with the spatial bundle metric ${}^{(n)}h$ induced by h:

Let X be any vector field and Y, Z be spacelike vector fields. There exist one-forms α, β such that

$$Y = h(\alpha, \cdot),\ Z = h(\beta, \cdot), \tag{1.12}$$

via which the induced metric is defined by ${}^{(n)}h(Y,Z) = h(\alpha,\beta) = \alpha(Z) = \beta(Y)$. For an arbitrary one-form κ, (1.12) means that $\kappa(Y) = h(\alpha, \kappa)$. Therefore, we can conclude

$$\begin{aligned}
\kappa\left(\overset{(n)}{\nabla}_X Y\right) &= \kappa(\nabla_X Y) \\
&= X(\kappa(Y)) - (\nabla_X \kappa)(Y) \\
&= X(h(\alpha,\kappa)) - h(\alpha, \nabla_X \kappa) \\
&= h(\nabla_X \alpha, \kappa),
\end{aligned} \tag{1.13}$$

where in the last step we used $\nabla h = 0$. We thus have shown that

$$\overset{(n)}{\nabla}_X Y = h(\nabla_X \alpha, \cdot). \tag{1.14}$$

Thus, we obtain

$$\begin{aligned}
X\left({}^{(n)}h(Y,Z)\right) &= X(\alpha(Z)) \\
&= (\nabla_X \alpha)(Z) + \alpha(\nabla_X Z) \\
&= (\nabla_X \alpha)(Z) + \alpha\left(\overset{(n)}{\nabla}_X Z\right) \\
&= {}^{(n)}h\left(\overset{(n)}{\nabla}_X Y, Z\right) + {}^{(n)}h\left(Y, \overset{(n)}{\nabla}_X Z\right),
\end{aligned} \tag{1.15}$$

which is compatibility of $\overset{(n)}{\nabla}$ with ${}^{(n)}h$.

Corollary 1.8. *On a Galilei manifold (M, τ, h) with integrable spacelike distribution, i.e. satisfying $\tau \wedge \mathrm{d}\tau = 0$, any Galilei connection ∇ induces a metric connection $\overset{(n)}{\nabla}$ on each of the spatial leaves (which are Riemannian manifolds). In the case of a torsion-free Galilei connection on a Galilei manifold with absolute time, the induced connections $\overset{(n)}{\nabla}$ are the Levi-Civita connections of the spatial leaves.* □

1.2.2. Classification

We now want to classify Galilei connections on a given Galilei manifold (M, τ, h). For this classification, we fix a vector field v (perhaps only locally defined) that is unit timelike future-directed[4], i.e. satisfies $\tau(v) = 1$. This field we call the *reference vector field*, with respect to which we will perform the classification. (Later, we will see that a possible interpretation of such a vector field is that it gives the directions of reference observers at each point, but this interpretation will be of no importance for now.)

[4]We can choose *any* such vector field, the choice of course being highly non-unique. That such a field always exists can be seen by, for example, choosing any Riemannian metric on M and using this to dualise τ into a vector field.

Definition 1.9. The *projector onto space along v* is the endomorphism of TM defined by

$$P^\mu_\nu := \delta^\mu_\nu - v^\mu \tau_\nu \ , \tag{1.16a}$$

or, in index-free notation,

$$P = \mathrm{id} - v \otimes \tau. \tag{1.16b}$$

Note that even though this projector depends on the choice of v, we will not acknowledge that in the notation, simply to avoid cluttering equations.

One easily shows that P^μ_ν is indeed a projector, its image is the space of spacelike vectors, and its kernel is pointwise spanned by v, such that the name given to it makes sense.

With respect to v, we can define a kind of inverse to h:

Definition 1.10. The *covariant space metric with respect to v* is the symmetric covariant tensor field $\underset{v}{h} = h_{\mu\nu}\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \in \Gamma(T^*M \otimes T^*M)$ defined by

$$h_{\mu\nu} v^\nu = 0,\ h_{\mu\nu} h^{\nu\rho} = P^\rho_\mu \ . \tag{1.17}$$

Note that in index notation, we leave out the subscript v in order to avoid confusion with an index. An h with indices 'downstairs' will always mean the covariant space metric with respect to that reference vector field v which is clear from context.

Existence and uniqueness of $\underset{v}{h}$ we can see as follows: at each point $p \in M$, the tangent and cotangent spaces decompose as

$$T_p M = \mathrm{span}\{v|_p\} \oplus \ker \tau|_p \ , \tag{1.18a}$$

$$T^*_p M = \mathrm{span}\{\tau|_p\} \oplus \ker v|_p \ . \tag{1.18b}$$

Therefore, under the map $h|_p \colon T^*_p M \to T_p M$, each (spacelike) vector in $\ker \tau|_p$ has a unique preimage in $\ker v|_p$ (compare construction 1.2). Now due to (1.17), the map $\underset{v}{h}|_p \colon T_p M \to T^*_p M$ needs to vanish on $\mathrm{span}\{v|_p\}$, its image has to lie inside $\ker v|_p$, and on $\ker \tau|_p$, it has to map each vector to its preimage under $h|_p$ in $\ker v|_p$. This fixes $\underset{v}{h}|_p$ uniquely.

Note that this also implies that the metric ${}^{(n)}h$ induced by h on spacelike vectors may be expressed as ${}^{(n)}h(w, \tilde{w}) = \underset{v}{h}(w, \tilde{w})$ for any two spacelike vectors $w, \tilde{w} \in \ker \tau|_p$, i.e. we have ${}^{(n)}h = \underset{v}{h}|_{\ker \tau}$.

Construction 1.11. The decomposition of vector fields and one-forms on M according to the above decomposition of the tangent and cotangent spaces induced by v we call the decomposition into *timelike* and *spacelike parts with respect to v*. Explicitly, for a vector field X and a one-form α, these decompositions are

$$X^\mu = \delta^\mu_\nu X^\nu = v^\mu \tau_\nu X^\nu + P^\mu_\nu X^\nu, \tag{1.19a}$$

$$\alpha_\mu = \delta^\nu_\mu \alpha_\nu = \tau_\mu v^\nu \alpha_\nu + P^\nu_\mu \alpha_\nu \ , \tag{1.19b}$$

where the first term is the timelike and the second term the spacelike part with respect to v. We may also apply this decomposition to higher-degree tensors / tensor fields.

Notation 1.12. In index notation for tensor fields on a Galilei manifold, when a unit timelike reference vector field v is chosen, we will lower indices using $\underset{v}{h}$. For example, given a tensor field with components $X^{\mu}{}_{\nu\rho}$ we will write

$$X_{\mu\nu\rho} := h_{\mu\lambda} X^{\lambda}{}_{\nu\rho} \,. \tag{1.20}$$

Note that first lowering an index (with $\underset{v}{h}$) and then raising it again (with h), or vice versa, corresponds to contracting it with the projector P along v. Therefore, we have to keep in mind the original position of indices in order not to forget these projections (except for spacelike indices, for which lowering and raising are actual inverses of each other).

Notation 1.13. For total symmetrisation and antisymmetrisation of tensors, we will in the following employ the common notational conventions of enclosing the respective indices in round or square brackets, respectively; e.g.

$$X_{(\mu\nu)} = \frac{1}{2}(X_{\mu\nu} + X_{\nu\mu}), \tag{1.21a}$$

$$X^{[\mu}{}_{\rho}{}^{\nu]} = \frac{1}{2}(X^{\mu}{}_{\rho}{}^{\nu} - X^{\nu}{}_{\rho}{}^{\mu}). \tag{1.21b}$$

Enclosed indices which are *not* to be (anti-)symmetrised will be surrounded by straight lines, such that for example in the expression $X^{(\mu\nu}h^{|\rho|\sigma)}$, only μ, ν, σ are symmetrised.

Definition 1.14. Consider a Galilei connection ∇ on a Galilei manifold (M, τ, h). The *Newton–Coriolis form*[5] of ∇ with respect to the unit timelike reference vector field v is the two-form Ω defined by

$$\Omega_{\mu\nu} := 2\big(\nabla_{[\mu} v^{\lambda}\big) h_{\nu]\lambda} \,. \tag{1.22}$$

Note that the index λ of $\nabla_\mu v^\lambda$ is spacelike (due to $\nabla\tau = 0$ and $\nabla_\mu 1 = 0$), such that lowering it with $h_{\nu\lambda}$ keeps all information. The antisymmetrisation however leads to Ω encoding less information than ∇v.

For the classification of Galilei connections, we will need an explicit form for $\nabla \underset{v}{h}$.

Lemma 1.15. *Consider a Galilei manifold (M, τ, h), and a unit timelike reference vector field v and a Galilei connection ∇ on it. Writing $\nabla_\mu v^\nu =: \Lambda_\mu{}^\nu$, the covariant derivative of $\underset{v}{h}$ is*

$$\nabla_\rho h_{\mu\nu} = -2\Lambda_{\rho(\mu}\tau_{\nu)} \,. \tag{1.23}$$

[5] This name was introduced by Geracie *et al.* in 2015 [GPR15], after previous authors had either given the form no name or called it just the 'Coriolis form'. Since, as we will see later, in a certain sense it contains the Newtonian potential (and not just the effects of Coriolis forces), they decided to include the name 'Newton'.

Proof. We decompose $\nabla \underset{v}{h}$ into its spacelike and its timelike part, namely

$$\begin{aligned}\nabla_\rho h_{\mu\nu} &= \delta^\sigma_\mu \nabla_\rho h_{\sigma\nu} \\ &= (P^\sigma_\mu + v^\sigma \tau_\mu)\nabla_\rho h_{\sigma\nu} \\ &= h_{\mu\lambda} h^{\lambda\sigma} \nabla_\rho h_{\sigma\nu} + \tau_\mu v^\sigma \nabla_\rho h_{\sigma\nu} \,. \end{aligned} \tag{1.24}$$

The timelike part we obtain from

$$0 = \nabla_\rho (v^\sigma h_{\sigma\nu}) = \Lambda_\rho{}^\sigma h_{\sigma\nu} + v^\sigma \nabla_\rho h_{\sigma\nu} = \Lambda_{\rho\nu} + v^\sigma \nabla_\rho h_{\sigma\nu} \,, \tag{1.25a}$$

and the spacelike one from

$$0 = \nabla_\rho \delta^\lambda_\nu = \nabla_\rho (P^\lambda_\nu + v^\lambda \tau_\nu) = \nabla_\rho (h^{\lambda\sigma} h_{\sigma\nu}) + \Lambda_\rho{}^\lambda \tau_\nu = h^{\lambda\sigma} \nabla_\rho h_{\sigma\nu} + \Lambda_\rho{}^\lambda \tau_\nu \,, \tag{1.25b}$$

where we have used $\nabla\tau = 0$, $\nabla h = 0$. Inserting these into (1.24), we obtain (1.23). □

Theorem 1.16 (Classification of Galilei connections)**.** *Let (M, τ, h) be a Galilei manifold and v a unit timelike reference vector field on it.*

(i) Let ∇ be a Galilei connection on (M, τ, h), and let Ω be the Newton–Coriolis form of ∇ with respect to v. Then the connection coefficients of ∇ take the form

$$\Gamma^\rho_{\mu\nu} = v^\rho \partial_{(\mu}\tau_{\nu)} + \frac{1}{2} h^{\rho\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) + \frac{1}{2} T^\rho{}_{\mu\nu} - T_{(\mu\nu)}{}^\rho + \tau_{(\mu}\Omega_{\nu)}{}^\rho . \tag{1.26}$$

(ii) Conversely, given any *two-form Ω and any tensor field T satisfying*

$$T^\rho{}_{\mu\nu} = -T^\rho{}_{\nu\mu} \,, \quad \tau_\rho T^\rho{}_{\mu\nu} = (\mathrm{d}\tau)_{\mu\nu} \,, \tag{1.27}$$

equation (1.26) defines a Galilei connection on (M, τ, h), whose torsion and Newton–Coriolis form with respect to v are the given T and Ω, respectively.

Proof. (i) For any three vector fields X, Y, Z, by applying the Leibniz rule and the definition of the torsion we have

$$\begin{aligned}\underset{v}{h}(\nabla_X Y, Z) &= X\big(\underset{v}{h}(Y,Z)\big) - \big(\nabla_X \underset{v}{h}\big)(Y,Z) - \underset{v}{h}(Y, \underbrace{\nabla_X Z}_{=\nabla_Z X + [X,Z] + T(X,Z)}) \\ &= X\big(\underset{v}{h}(Y,Z)\big) - \big(\nabla_X \underset{v}{h}\big)(Y,Z) - \underset{v}{h}(Y, \nabla_Z X) - \underset{v}{h}(Y, T(X,Z) + [X,Z]). \end{aligned} \tag{1.28a}$$

Cyclicly permuting the vector fields, from this we obtain

$$\underset{v}{h}(\nabla_Y Z, X) = Y\big(\underset{v}{h}(Z,X)\big) - \big(\nabla_Y \underset{v}{h}\big)(Z,X) - \underset{v}{h}(Z, \nabla_X Y) - \underset{v}{h}(Z, T(Y,X) + [Y,X]), \tag{1.28b}$$

$$\underset{v}{h}(\nabla_Z X, Y) = Z\big(\underset{v}{h}(X,Y)\big) - \big(\nabla_Z \underset{v}{h}\big)(X,Y) - \underset{v}{h}(X, \nabla_Y Z) - \underset{v}{h}(X, T(Z,Y) + [Z,Y]). \tag{1.28c}$$

Considering (1.28a) + (1.28b) − (1.28c) now yields

$$\begin{aligned} 2\underset{v}{h}(\nabla_X Y, Z) = {} & X\big(\underset{v}{h}(Y,Z)\big) + Y\big(\underset{v}{h}(Z,X)\big) - Z\big(\underset{v}{h}(X,Y)\big) \\ & - \big(\nabla_X \underset{v}{h}\big)(Y,Z) - \big(\nabla_Y \underset{v}{h}\big)(Z,X) + \big(\nabla_Z \underset{v}{h}\big)(X,Y) \\ & - \underset{v}{h}(Y, T(X,Z) + [X,Z]) - \underset{v}{h}(Z, T(Y,X) + [Y,X]) \\ & + \underset{v}{h}(X, T(Z,Y) + [Z,Y]). \end{aligned} \tag{1.29}$$

Applied to the coordinate vector fields $X = \partial_\mu$, $Y = \partial_\nu$, $Z = \partial_\sigma$, this gives

$$\begin{aligned} h_{\sigma\lambda}\Gamma^\lambda_{\mu\nu} = {} & \frac{1}{2}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}) + \Lambda_{\mu(\nu}\tau_{\sigma)} + \Lambda_{\nu(\sigma}\tau_{\mu)} - \Lambda_{\sigma(\mu}\tau_{\nu)} \\ & + \frac{1}{2}(-T_{\nu\mu\sigma} - T_{\sigma\nu\mu} + T_{\mu\sigma\nu}). \end{aligned} \tag{1.30}$$

Here we have used lemma 1.15 to express $\nabla \underset{v}{h}$ in terms of ∇v, again writing $\nabla_\mu v^\nu =: \Lambda_\mu{}^\nu$.

Equation (1.30) gives us an expression for the spacelike part of the connection coefficients. The timelike part we obtain from the compatibility $\nabla\tau = 0$, which in components reads $0 = \partial_\mu \tau_\nu - \Gamma^\lambda_{\mu\nu}\tau_\lambda$. Combining those, we obtain

$$\begin{aligned} \Gamma^\rho_{\mu\nu} &= (v^\rho \tau_\lambda + h^{\rho\sigma} h_{\sigma\lambda})\Gamma^\lambda_{\mu\nu} \\ &= v^\rho \partial_\mu \tau_\nu + \frac{1}{2}h^{\rho\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) + \frac{1}{2}h^{\rho\sigma}(T_{\sigma\mu\nu} - T_{\nu\mu\sigma} - T_{\mu\nu\sigma}) \\ &\quad + h^{\rho\sigma}(\Lambda_{\mu(\nu}\tau_{\sigma)} + \Lambda_{\nu(\sigma}\tau_{\mu)} - \Lambda_{\sigma(\nu}\tau_{\mu)}) \\ &= v^\rho \partial_\mu \tau_\nu + \frac{1}{2}h^{\rho\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) + \frac{1}{2}P^\rho_\lambda T^\lambda{}_{\mu\nu} - T_{(\mu\nu)}{}^\rho \\ &\quad + h^{\rho\sigma}\big(\Lambda_{(\mu|\sigma|}\tau_{\nu)} - \Lambda_{\sigma(\mu}\tau_{\nu)}\big)\,. \end{aligned} \tag{1.31}$$

Considering the first torsion term, we note that due to the temporal torsion being $\mathrm{d}\tau$ (proposition 1.6 (i)), we can write

$$\frac{1}{2}P^\rho_\lambda T^\lambda{}_{\mu\nu} = \frac{1}{2}T^\rho{}_{\mu\nu} - \frac{1}{2}v^\rho \underbrace{\tau_\lambda T^\lambda{}_{\mu\nu}}_{=(\mathrm{d}\tau)_{\mu\nu} = 2\partial_{[\mu}\tau_{\nu]}} = \frac{1}{2}T^\rho{}_{\mu\nu} - \partial_{[\mu}\tau_{\nu]}\,. \tag{1.32}$$

Finally, the definition of the Newton–Coriolis form is $\Omega_{\mu\sigma} = 2\Lambda_{[\mu\sigma]} = \Lambda_{\mu\sigma} - \Lambda_{\sigma\mu}$, such that we may rewrite

$$h^{\rho\sigma}(\Lambda_{(\mu|\sigma|}\tau_{\nu)} - \Lambda_{\sigma(\mu}\tau_{\nu)}) = h^{\rho\sigma}\Omega_{(\mu|\sigma|}\tau_{\nu)} = \Omega_{(\mu}{}^{\rho}\tau_{\nu)} = \tau_{(\mu}\Omega_{\nu)}{}^{\rho}. \tag{1.33}$$

Inserting (1.32) and (1.33) into (1.31), we obtain the connection coefficients as stated in the theorem.

(ii) The proof that (1.26) defines a Galilei connection is covered in detail in an exercise on the first exercise sheet. Its torsion being the given T is obvious (consider $2\Gamma^{\rho}_{[\mu\nu]}$). That its Newton–Coriolis form is the given Ω requires some calculation, which we also leave as an exercise. □

Corollary 1.17. *On a Galilei manifold with absolute time, the torsion-free Galilei connections are classified by their respective Newton–Coriolis forms (with respect to a choice of unit timelike reference vector field).* □

Note that this is different from the case of (pseudo-)Riemannian manifolds, where the metric-compatible connections are classified by their torsion and therefore there is a unique torsion-free connection, the Levi-Civita connection.

Definition 1.18. Any change from one unit timelike reference vector field to another has the form

$$v^\mu \to \tilde{v}^\mu = v^\mu - k^\mu \tag{1.34}$$

for a spacelike vector field k. Such a change of reference field is called a *Milne boost*.

Proposition 1.19. *Under a Milne boost (1.34), the projector P, the covariant space metric $\underset{v}{h}$ and the Newton–Coriolis form (of some fixed Galilei connection) with respect to the reference vector field transform according to*

$$P \to \tilde{P} = P + k \otimes \tau, \tag{1.35a}$$

$$\underset{v}{h} \to \underset{\tilde{v}}{h} = \underset{v}{h} + k^\flat \otimes \tau + \tau \otimes k^\flat + k^2 \tau \otimes \tau, \tag{1.35b}$$

$$\Omega \to \tilde{\Omega} = \Omega - \mathrm{d}k^\flat - \frac{1}{2}\mathrm{d}(k^2) \wedge \tau + \underset{v}{h}(k, T(\cdot,\cdot)), \tag{1.35c}$$

where $k^\flat = \underset{v}{h}(k,\cdot)$ is the one-form associated to k via $\underset{v}{h}$, and $k^2 = {}^{(n)}h(k,k)$ is the squared length of k. Written in components, the above formulae read

$$P^\mu_\nu \to P^\mu_\nu + k^\mu \tau_\nu\ , \tag{1.36a}$$

$$h_{\mu\nu} \to h_{\mu\nu} + k_\mu \tau_\nu + \tau_\mu k_\nu + k^2 \tau_\mu \tau_\nu\ , \tag{1.36b}$$

$$\Omega_{\mu\nu} \to \Omega_{\mu\nu} - 2\partial_{[\mu}k_{\nu]} - (\partial_{[\mu}k^2)\tau_{\nu]} + k_\rho T^\rho{}_{\mu\nu}\ , \tag{1.36c}$$

where the index on k_μ has been lowered with $\underset{v}{h}$.

Proof. The transformation behaviour of the projector follows directly from its definition. For the covariant space metric and the Newton–Coriolis form, some calculation has to be done, which we leave as an exercise (see exercise sheet 2). □

At this point, the transformation behaviour of $\underset{v}{h}$ and Ω under Milne boosts seems to appear 'out of the blue'. Later, we will see how to actually *derive* it.

Finally, we want to introduce bases of the tangent space that are adapted to the structure of a Galilei manifold.

Definition 1.20. Let (M, τ, h) be a Galilei manifold. A *Galilei basis* at a point $p \in M$ is an (ordered) basis $(\mathrm{e}_A) = (\mathrm{e}_t, \mathrm{e}_a)$, $a = 1, \ldots, n$, of the tangent space T_pM such that

(i) for the dual basis $(\mathrm{e}^A) = (\mathrm{e}^t, \mathrm{e}^a)$ of T_p^*M, we have $\mathrm{e}^t = \tau|_p$, and

(ii) $h|_p = \delta^{ab}\mathrm{e}_a \otimes \mathrm{e}_b$, i.e. in components $h^{\mu\nu}|_p = \delta^{ab}\mathrm{e}_a^\mu \mathrm{e}_b^\nu$.

By definition of a Galilei manifold, the second condition implies that the vectors e_a are spacelike: we have $0 = h|_p(\tau, \cdot) = \delta^{ab}\tau(\mathrm{e}_a)\mathrm{e}_b$, which implies $\tau(\mathrm{e}_a) = 0$.

Proposition 1.21. *In the definition of a Galilei basis, condition (i) may be replaced by the following:*

(i') e_t is a future-directed unit timelike vector.

Similarly, as long as (i) is kept, condition (ii) may be replaced by the following:

(ii') The 'spacelike' elements e^a of the dual basis $(\mathrm{e}^A) = (\mathrm{e}^t, \mathrm{e}^a)$ are orthonormal with respect to h, i.e. we have $h(\mathrm{e}^a, \mathrm{e}^b) = \delta^{ab}$.

Proof. Given a basis satisfying the original conditions, by the definition of the dual basis, we have $\tau(\mathrm{e}_t) = \mathrm{e}^t(\mathrm{e}_t) = 1$, i.e. e_t is future-directed unit timelike. Conversely, given (i') and (ii), we have $\tau(\mathrm{e}_t) = 1$ and $\tau(\mathrm{e}_a) = 0$, which implies $\tau|_p = \mathrm{e}^t$, i.e. (i).

Now for (ii'). Given a basis satisfying (i) and (ii), the definition of the dual basis gives us $h(\mathrm{e}^a, \mathrm{e}^b) = \delta^{cd}(\mathrm{e}_c \otimes \mathrm{e}_d)(\mathrm{e}^a, \mathrm{e}^b) = \delta^{cd}\delta_c^a\delta_d^b = \delta^{ab}$. Conversely, given (i) and (ii'), we have $0 = h(\tau|p, \cdot) = h(\mathrm{e}^t, \cdot)$, such that h is given by $h|_p = h(\mathrm{e}^a, \mathrm{e}^b)e_a \otimes \mathrm{e}_b = \delta^{ab}e_a \otimes \mathrm{e}_b$. □

Note that together with $\mathrm{e}^t = \tau|_p$, condition (ii') means that the dual basis is a Sylvester basis for $h|_p$. Also note that (i') and (ii') together are *weaker* than the original conditions, i.e. they do *not* characterise a Galilei basis.

Also directly from the definition of the dual basis, we obtain the following:

Proposition 1.22. *Let (M, τ, h) be a Galilei manifold and (e_A) a Galilei basis at $p \in M$.*

(i) *The projector onto space*[6] *along* e_t *can be expressed as*

$$P^\mu_\nu = \mathrm{e}^\mu_a \mathrm{e}^a_\nu \tag{1.37a}$$

or, in index-free notation,

$$P = \mathrm{e}_a \otimes \mathrm{e}^a. \tag{1.37b}$$

(ii) *The covariant space metric with respect to* e_t *can be expressed as*

$$h_{\mu\nu} = \delta_{ab}\mathrm{e}^a_\mu \mathrm{e}^b_\nu \tag{1.38a}$$

or, in index-free notation,

$$\underset{\mathrm{e}_t}{h} = \delta_{ab}\mathrm{e}^a \otimes \mathrm{e}^b. \tag{1.38b}$$

Proof. (i) The definition of the dual basis can be expressed in the form $\mathrm{id}_{T_pM} = \mathrm{e}_A \otimes \mathrm{e}^A = \mathrm{e}_t \otimes \mathrm{e}^t + \mathrm{e}_a \otimes \mathrm{e}^a$. By the definition of the projector, the result follows.

(ii) Contracting $\delta_{ab}\mathrm{e}^a_\mu \mathrm{e}^b_\nu$ with e^μ_t and with $h^{\nu\rho}$, we obtain

$$\delta_{ab}\mathrm{e}^a_\mu \mathrm{e}^b_\nu \mathrm{e}^\mu_t = 0 \tag{1.39}$$

and

$$\delta_{ab}\mathrm{e}^a_\mu \mathrm{e}^b_\nu h^{\nu\rho} = \delta_{ab}\mathrm{e}^a_\mu \mathrm{e}^b_\nu \delta^{cd}\mathrm{e}^\nu_c \mathrm{e}^\rho_d = \delta_{ab}\mathrm{e}^a_\mu \delta^b_c \delta^{cd}\mathrm{e}^\rho_d = \mathrm{e}^a_\mu \mathrm{e}^\rho_a = P^\rho_\mu\,, \tag{1.40}$$

i.e. we have shown the defining properties of the covariant space metric. □

1.2.3. Newtonian connections

Consider a Galilei manifold with absolute time, and a torsion-free Galilei connection on it. We want to count the number of algebraically independent components of the connection's curvature tensor (at any point), and compare this to the (pseudo-)Riemannian case.

Recall that for the Levi-Civita connection of an n-dimensional (pseudo-)Riemannian manifold, the (Riemannian) curvature tensor has $\frac{n^2(n^2-1)}{12}$ independent components. Since the curvature tensor of a Galilei connection has less symmetries than in the Riemannian case, it has more independent components:

[6] We have only defined this projector for unit timelike vector *fields*, but of course the definition also works just at a single point. The same goes for the covariant space metric.

Proposition 1.23. *Let (M, τ, h) be an $(n+1)$-dimensional Galilei manifold with absolute time, and ∇ a torsion-free Galilei connection on it. At any point of M, the number of algebraically independent components of the curvature tensor of ∇ is*

$$\frac{n^2(n+1)(n+5)}{12} = \frac{(n+1)^2((n+1)^2-1)}{12} + \frac{n(n^2-1)}{6}, \tag{1.41}$$

i.e. $\frac{n(n^2-1)}{6}$ more than that for an $(n+1)$-dimensional (pseudo-)Riemannian manifold.

Proof. The first index of the curvature tensor is spacelike (proposition 1.6 (ii)),

$$\tau_\mu R^\mu{}_{\nu\rho\sigma} = 0. \tag{1.42a}$$

Furthermore, the curvature tensor has the following symmetries: it is antisymmetric in its last two indices,

$$R^\mu{}_{\nu\rho\sigma} = -R^\mu{}_{\nu\sigma\rho}\,, \tag{1.42b}$$

antisymmetric in its first two indices when the second is raised (proposition 1.6 (ii)),

$$R^{\mu\nu}{}_{\rho\sigma} = -R^{\nu\mu}{}_{\rho\sigma}\,, \tag{1.42c}$$

and due to vanishing torsion satisfies the Bianchi identity

$$R^\mu{}_{[\nu\rho\sigma]} = 0. \tag{1.42d}$$

Due to antisymmetry in the last two indices, the Bianchi identity can also be stated as the vanishing of the cyclic sum over the last three indices,

$$R^\mu{}_{\nu\rho\sigma} + R^\mu{}_{\rho\sigma\nu} + R^\mu{}_{\sigma\nu\rho} = 0. \tag{1.42e}$$

We now focus on a single point $p \in M$ and want to analyse these symmetries in a Galilei basis $(\mathrm{e}_A) = (\mathrm{e}_t, \mathrm{e}_a)$ of T_pM. The first three symmetries (1.42) of the curvature then take the form

$$R^t{}_{\nu\rho\sigma} = 0, \tag{1.43a}$$

$$R_{a\nu\rho\sigma} = R_{a\nu\rho\sigma}\,, \tag{1.43b}$$

$$R_{ab\rho\sigma} = -R_{ba\rho\sigma}\,, \tag{1.43c}$$

where we have lowered the first index with δ_{ab} when it is spacelike.

The purely spatial part R_{abcd} of the curvature tensor now has precisely the symmetries of the curvature tensor of an n-dimensional Riemannian manifold (in fact, it *is* the Riemannian curvature of the spatial leaves), so it has $\frac{n^2(n^2-1)}{12}$ independent components.

(Note that the 'symmetry in pairs' $R_{abcd} = R_{cdab}$ follows from the Bianchi identity and the antisymmetries.)

Thus, we still need to count the number of independent mixed temporal-spatial components of the curvature, which have the form $R_{a\nu\rho\sigma} = -R_{a\nu\sigma\rho}$ with at least one of the indices ν, ρ, σ being t. On the one hand, this includes components of the form $R_{at\rho\sigma}$. For those, we have n independent choices for the value of a, and $\frac{(n+1)n}{2}$ choices for the antisymmetric pair $\rho\sigma$; thus these are $n \cdot \frac{(n+1)n}{2}$ independent components. On the other hand, we have the components R_{abtc}. Here for the antisymmetric pair ab, we have $\frac{n(n-1)}{2}$ choices, and n choices for c, meaning those give us $\frac{n(n-1)}{2} \cdot n$ independent components.

In the above counting, we have however not yet taken the Bianchi identity into account, which tells us that $R_{at\rho\sigma} + (\text{cyclic terms in } t, \rho, \sigma) = 0$. This is a non-trivial requirement (i.e. one which does not follow from the already considered symmetries) for either all four indices being different, or for one of ρ, σ being a. The former case gives $n \cdot \binom{n-1}{2}$ equations (n choices for a, then two values ρ, σ from the remaining $n-1$); the latter amounts to $0 = R_{atab} + \cancelto{0}{R_{aabt}} + R_{abta}$, which are $n(n-1)$ independent equations.

Combining everything, the number of independent components of the curvature tensor at p is

$$\underbrace{\frac{n^2(n^2-1)}{12}}_{R_{abcd}} + \underbrace{n \cdot \frac{(n+1)n}{2}}_{R_{at\rho\sigma}} + \underbrace{\frac{n(n-1)}{2} \cdot n}_{R_{abtc}} \underbrace{- n \cdot \binom{n-1}{2} - n(n-1)}_{\text{Bianchi}}, \tag{1.44}$$

which by a little calculation can be shown to be equal to the number given in (1.41). □

We now want to introduce a requirement on the curvature that reduces the number of independent components to as many as in the (pseudo)-Riemannian case. The only symmetry holding in the (pseudo-)Riemannian case but absent above was the 'symmetry in pairs' for the mixed temporal-spatial curvature components.

Definition 1.24. A *Newtonian connection* on a Galilei manifold (M, τ, h) with absolute time is a torsion-free Galilei connection whose curvature tensor satisfies the additional symmetry

$$R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma} = R^{\nu}{}_{\sigma}{}^{\mu}{}_{\rho} \,. \tag{1.45}$$

A *Newtonian manifold* is a Galilei manifold with absolute time with a Newtonian connection on it.

Theorem 1.25. *The number of algebraically independent components of the curvature tensor of an $(n+1)$-dimensional Newtonian manifold is*

$$\frac{(n+1)^2((n+1)^2-1)}{12}, \tag{1.46}$$

as for an $(n+1)$-dimensional (pseudo-)Riemannian manifold.

Proof. In a Galilei basis, the additional symmetries introduced by the condition of being Newtonian read

$$R_{atbt} = R_{btat} \text{ for } a \neq b, \tag{1.47a}$$
$$R_{atbc} = R_{bcat} \text{ for } a, b, c \text{ all different} \tag{1.47b}$$

(for $a = b$, the second one follows from the Bianchi identity). The first case gives us $\binom{n}{2}$ equations, and the second one $\binom{n}{3}$. Together, these are

$$\binom{n}{2} + \binom{n}{3} = \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{6} = \frac{n(n-1)(n+1)}{6} = \frac{n(n^2-1)}{6} \tag{1.48}$$

additional requirements for the curvature, which reduces the number of components from (1.41) to the (pseudo-)Riemannian value. □

The condition of a Galilei connection being Newtonian can be related to its Newton–Coriolis form:

Lemma 1.26. *Let (M, τ, h) be a Galilei manifold with absolute time, ∇ be a torsion-free Galilei connection on it, and Ω be its Newton–Coriolis form with respect to any unit timelike reference vector field. Then the curvature tensor of ∇ satisfies*

$$R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma} - R^{\nu}{}_{\sigma}{}^{\mu}{}_{\rho} = (\mathrm{d}\Omega)^{\mu\nu}{}_{(\rho}\tau_{\sigma)} \ . \tag{1.49}$$

Proof. We denote the reference vector field by v, and its covariant derivative by $\nabla v =: \Lambda$. Then the definition of Ω reads $\Omega_{\mu\nu} = 2\Lambda_{[\mu\nu]}$. Due to torsion-freeness, we can thus express the exterior derivative as

$$\begin{aligned}
(\mathrm{d}\Omega)_{\mu\nu\rho} &= 3\,\partial_{[\mu}\Omega_{\nu\rho]} \\
&= 3(\nabla_{[\mu}\Omega_{\nu\rho]} + \underbrace{\Gamma^{\lambda}_{[\mu\nu}}_{=\frac{1}{2}T^{\lambda}{}_{[\mu\nu}=0}\Omega_{|\lambda|\rho]} + \underbrace{\Gamma^{\lambda}_{[\mu\rho}}_{=\frac{1}{2}T^{\lambda}{}_{[\mu\rho}=0}\Omega_{\nu]\lambda}) \\
&= 6\,\nabla_{[\mu}\Lambda_{\nu\rho]} \ .
\end{aligned} \tag{1.50}$$

Now in general, for any torsion-free connection and any one-form α, we have the identity

$$2\nabla_{[\mu}\nabla_{\nu]}\alpha_\rho = -R^\lambda{}_{\rho\mu\nu}\alpha_\lambda \tag{1.51}$$

(with torsion, we would have an additional $-T^\lambda{}_{\mu\nu}\nabla_\lambda\alpha_\rho$ on the right-hand side). Applying this to a tensor product $\alpha_\rho\beta_\sigma$, we obtain

$$\begin{aligned} 2\nabla_{[\mu}\nabla_{\nu]}(\alpha_\rho\beta_\sigma) &= 2(\nabla_{[\mu}\nabla_{\nu]}\alpha_\rho)\beta_\sigma + 2\alpha_\rho\nabla_{[\mu}\nabla_{\nu]}\beta_\sigma \\ &= -R^\lambda{}_{\rho\mu\nu}\alpha_\lambda\beta_\sigma - R^\lambda{}_{\sigma\mu\nu}\alpha_\rho\beta_\lambda \,. \end{aligned} \tag{1.52}$$

Since the covariant space metric $h_{\rho\sigma}$ may be decomposed as a sum of tensor products of one-forms, we obtain

$$\nabla_{[\mu}\nabla_{\nu]}h_{\rho\sigma} = -h_{\lambda(\rho}R^\lambda{}_{\sigma)\mu\nu} \,, \tag{1.53}$$

which with lemma 1.15 takes the form

$$R_{(\rho\sigma)\mu\nu} = 2\nabla_{[\mu}\Lambda_{\nu](\rho}\tau_{\sigma)} \,. \tag{1.54}$$

Contracting this with $6v^\sigma$ and antisymmetrising in ρ, μ, ν, we obtain

$$3R_{[\rho|\sigma|\mu\nu]}v^\sigma = 6\nabla_{[\mu}\Lambda_{\nu\rho]} + 6\tau_{[\rho}\nabla_\mu\Lambda_{\nu]\sigma}v^\sigma. \tag{1.55}$$

Due to

$$\begin{aligned} \nabla_{[\mu}\Lambda_{\nu]\sigma}v^\sigma &= \nabla_{[\mu}(\underbrace{\Lambda_{\nu]\sigma}v^\sigma}_{=\Lambda_{\nu]}{}^\kappa h_{\kappa\sigma}v^\sigma=0}) - \Lambda_{[\nu|\sigma|}\nabla_{\mu]}v^\sigma \\ &= -\Lambda_{[\nu|\sigma|}\Lambda_{\mu]}{}^\sigma \\ &= -\frac{1}{2}(\Lambda_{\nu\sigma}\Lambda_\mu{}^\sigma - \Lambda_{\mu\sigma}\Lambda_\nu{}^\sigma) = 0 \end{aligned} \tag{1.56}$$

and (1.50), we thus arrive at

$$\begin{aligned} (\mathrm{d}\Omega)_{\mu\nu\rho} &= 3R_{[\rho|\lambda|\mu\nu]}v^\lambda \\ &= (R_{\rho\lambda\mu\nu} + R_{\mu\lambda\nu\rho} + R_{\nu\lambda\rho\mu})v^\lambda, \end{aligned} \tag{1.57}$$

where we have used antisymmetry of the curvature in the last two indices to rewrite the antisymmetrisation as a cyclic sum. Now using the Bianchi identity, we can rewrite the first and second terms on the right-hand side of (1.57) as

$$R_{\rho\lambda\mu\nu} = -R_{\rho\mu\nu\lambda} - R_{\rho\nu\lambda\mu} \,, \tag{1.58a}$$
$$R_{\mu\lambda\nu\rho} = -R_{\mu\nu\rho\lambda} - R_{\mu\rho\lambda\nu} \,. \tag{1.58b}$$

By decomposing the second index of R into its space- and timelike parts, we obtain the identity

$$
\begin{aligned}
R_{(\mu\nu)\rho\sigma} &= (P^\kappa_{(\nu} + v^\kappa \tau_{(\nu})R_{\mu)\kappa\rho\sigma} \\
&= h_{\lambda(\mu}h_{|\kappa|\nu)}R^{\lambda\kappa}{}_{\rho\sigma} + v^\kappa\tau_{(\nu}R_{\mu)\kappa\rho\sigma} \\
&= h_{\lambda\mu}h_{\kappa\nu}\underbrace{R^{(\lambda\kappa)}{}_{\rho\sigma}}_{=0} + v^\kappa\tau_{(\nu}R_{\mu)\kappa\rho\sigma} \\
&= v^\kappa\tau_{(\mu}R_{\nu)\kappa\rho\sigma}\,,
\end{aligned} \tag{1.59}
$$

where we used antisymmetry of the spacelike part of the first two indices of the curvature tensor (proposition 1.6 (ii)). Using this identity, we can rewrite both terms on the right-hand side of (1.58a) and the first term on the right-hand side of (1.58b) as

$$
\begin{aligned}
-R_{\rho\mu\nu\lambda} &= R_{\mu\rho\nu\lambda} - 2v^\kappa\tau_{(\rho}R_{\mu)\kappa\nu\lambda}\,, && (1.60a)\\
-R_{\rho\nu\lambda\mu} &= R_{\nu\rho\lambda\mu} - 2v^\kappa\tau_{(\rho}R_{\nu)\kappa\lambda\mu}\,, && (1.60b)\\
-R_{\mu\nu\rho\lambda} &= R_{\nu\mu\rho\lambda} - 2v^\kappa\tau_{(\mu}R_{\nu)\kappa\rho\lambda}\,. && (1.60c)
\end{aligned}
$$

Using (1.58) and (1.60) in (1.57), we obtain

$$
\begin{aligned}
(\mathrm{d}\Omega)_{\mu\nu\rho} &= (R_{\mu\rho\nu\lambda} + \underbrace{R_{\nu\rho\lambda\mu} + R_{\nu\mu\rho\lambda}}_{=-R_{\nu\lambda\mu\rho}} - R_{\mu\rho\lambda\nu} + R_{\nu\lambda\rho\mu} \\
&\quad - 2v^\kappa\tau_{(\rho}R_{\mu)\kappa\nu\lambda} - 2v^\kappa\tau_{(\rho}R_{\nu)\kappa\lambda\mu} - 2v^\kappa\tau_{(\mu}R_{\nu)\kappa\rho\lambda})v^\lambda \\
&= (2R_{\mu\rho\nu\lambda} - 2R_{\nu\lambda\mu\rho} - 2v^\kappa\tau_{(\rho}R_{\mu)\kappa\nu\lambda} + 2v^\kappa\tau_{(\rho}R_{\nu)\kappa\mu\lambda} - 2v^\kappa\tau_{(\mu}R_{\nu)\kappa\rho\lambda})v^\lambda,
\end{aligned} \tag{1.61}
$$

where we have used the Bianchi identity and antisymmetry in the last two indices to combine some terms. Raising μ and ν in this equation, we get

$$
\begin{aligned}
(\mathrm{d}\Omega)^{\mu\nu}{}_\rho &= (2R^{\mu}{}_{\rho}{}^{\nu}{}_{\lambda} - 2R^{\nu}{}_{\lambda}{}^{\mu}{}_{\rho} - v^\kappa\tau_\rho R^{\mu}{}_{\kappa}{}^{\nu}{}_{\lambda} + v^\kappa\tau_\rho R^{\nu}{}_{\kappa}{}^{\mu}{}_{\lambda})v^\lambda \\
&= \underbrace{(R^{\mu}{}_{\kappa}{}^{\nu}{}_{\lambda} - R^{\nu}{}_{\lambda}{}^{\mu}{}_{\kappa})}_{=:M^{\mu}{}_{\kappa}{}^{\nu}{}_{\lambda}}(2\delta^\kappa_\rho - v^\kappa\tau_\rho)v^\lambda \\
&= M^{\mu}{}_{\kappa}{}^{\nu}{}_{\lambda}(\delta^\kappa_\rho + P^\kappa_\rho)v^\lambda.
\end{aligned} \tag{1.62}
$$

Due to antisymmetry of $\mathrm{d}\Omega$, from this we also obtain

$$
(\mathrm{d}\Omega)^{\mu\nu}{}_\sigma = -(\mathrm{d}\Omega)^{\nu\mu}{}_\sigma \overset{(1.62)}{=} -M^{\nu}{}_{\kappa}{}^{\mu}{}_{\lambda}(\delta^\kappa_\sigma + P^\kappa_\sigma)v^\lambda = M^{\mu}{}_{\lambda}{}^{\nu}{}_{\kappa}(\delta^\kappa_\sigma + P^\kappa_\sigma)v^\lambda = M^{\mu}{}_{\kappa}{}^{\nu}{}_{\lambda}(\delta^\lambda_\sigma + P^\lambda_\sigma)v^\kappa, \tag{1.63}
$$

where we have used the definition of M in the second last step and renamed contracted indices in the last step. Now multiplying (1.62) with τ_σ and (1.63) with τ_ρ, we get the two equations

$$(\mathrm{d}\Omega)^{\mu\nu}{}_\rho \tau_\sigma = M^{\mu}{}_{\kappa}{}^{\nu}{}_{\lambda}(\delta^\kappa_\rho + P^\kappa_\rho)(\delta^\lambda_\sigma - P^\lambda_\sigma), \tag{1.64a}$$
$$(\mathrm{d}\Omega)^{\mu\nu}{}_\sigma \tau_\rho = M^{\mu}{}_{\kappa}{}^{\nu}{}_{\lambda}(\delta^\kappa_\rho - P^\kappa_\rho)(\delta^\lambda_\sigma + P^\lambda_\sigma). \tag{1.64b}$$

Adding these and dividing by 2, we arrive at

$$\begin{aligned}(\mathrm{d}\Omega)^{\mu\nu}{}_{(\rho}\tau_{\sigma)} &= M^{\mu}{}_{\kappa}{}^{\nu}{}_{\lambda}(\delta^\kappa_\rho \delta^\lambda_\sigma - P^\kappa_\rho P^\lambda_\sigma) \\ &= M^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma} - M^{\mu\kappa\nu\lambda} h_{\rho\kappa} h_{\sigma\lambda}.\end{aligned} \tag{1.65}$$

Due to symmetry in pairs of the purely spacelike part of the curvature tensor, we have $M^{\mu\kappa\nu\lambda} = 0$, and the proof is finished. □

Theorem 1.27. *A torsion-free Galilei connection is Newtonian iff its Newton–Coriolis form with respect to any unit timelike reference vector field (and then all such fields) is closed.*

Proof. If $\mathrm{d}\Omega = 0$, then by lemma 1.26 the connection is Newtonian. Conversely, if it is Newtonian, we have $(\mathrm{d}\Omega)^{\mu\nu}{}_{(\sigma}\tau_{\rho)} = 0$. This implies $(\mathrm{d}\Omega)^{\mu\nu\sigma} = 0$, i.e. vanishing of the purely spacelike part of $\mathrm{d}\Omega$, and $(\mathrm{d}\Omega)^{\mu\nu}{}_\sigma v^\sigma = 0$ for the timelike reference field v. Due to the antisymmetry of $\mathrm{d}\Omega$, this implies $\mathrm{d}\Omega = 0$. □

Construction 1.28. According to construction 1.7 and corollary 1.8, a torsion-free Galilei connection ∇ on a Galilei manifold (M, τ, h) with absolute time restricts on each spatial leaf to the Levi-Civita connection $\overset{(n)}{\nabla}$ of the induced Riemannian metric ${}^{(n)}h$.

Therefore, we can restrict the curvature tensor of ∇ to each of the spatial leaves, and when doing so, we obtain the curvature tensor ${}^{(n)}R$ of the spatial leaf as a Riemannian manifold: for any spatial leaf Σ and vector fields $X, Y, Z \in \Gamma(T\Sigma)$, on Σ we have

$$\begin{aligned} R(X,Y)Z &= \nabla_X \underbrace{\nabla_Y Z}_{=\overset{(n)}{\nabla}_Y Z} - \nabla_Y \underbrace{\nabla_X Z}_{=\overset{(n)}{\nabla}_X Z} - \nabla_{[X,Y]} Z \\ &= \overset{(n)}{\nabla}_X \overset{(n)}{\nabla}_Y Z - \overset{(n)}{\nabla}_Y \overset{(n)}{\nabla}_X Z - \overset{(n)}{\nabla}_{[X,Y]} Z \\ &= {}^{(n)}R(X,Y)Z \end{aligned} \tag{1.66}$$

(where of course to be able to apply ∇, we have to extend X, Y, Z to vector fields in a neighbourhood of Σ in M in some way, the choice of which does not affect the results when restricted to Σ).

Furthermore, when restricting the Ricci tensor Ric of ∇ to any spatial leaf, we obtain the Ricci tensor ${}^{(n)}\mathrm{Ric}$ of the leaf as a Riemannian manifold: for any two spacelike vectors $v, w \in \ker \tau|_p$, we have[7]

$$\begin{aligned} {}^{(n)}\mathrm{Ric}(v,w) &= \operatorname{tr}_{\ker\tau|_p}\left({}^{(n)}R(\cdot,w)v\right) \\ &= \operatorname{tr}_{\ker\tau|_p}(R(\cdot,w)v) \\ (\tau_\mu R^\mu{}_{\nu\rho\sigma} = 0) \quad &= \operatorname{tr}_{T_pM}(R(\cdot,w)v) \\ &= \mathrm{Ric}(v,w). \end{aligned} \tag{1.67}$$

Definition 1.29. A Galilei manifold with absolute time is *spatially flat* iff each of the spatial leaves foliating M is flat (as a Riemannian manifold).

Note that due to the preceding construction, spatial flatness is equivalent to vanishing of the purely spatial part of the curvature tensor of any torsion-free Galilei connection, i.e. to the curvature satisfying

$$R^{\mu\nu\rho\sigma} = 0. \tag{1.68}$$

[7]In index notation in a Galilei basis, the calculation takes the form ${}^{(n)}R_{ab}v^a w^b = {}^{(n)}R^c{}_{acd}v^a w^b = R^c{}_{acb}v^a w^b = R^\mu{}_{a\mu b}v^a w^b = R_{\mu\nu}v^\mu w^\nu$.

2. Classical Newton–Cartan gravity

In this chapter, we will use the language of Galilei manifolds and connections developed previously to introduce and discuss 'classical' Newton–Cartan gravity, i.e. the version of the theory presented by Künzle [Kün76], Ehlers [Ehl81] and Malament [Mal12]. We will see how to recover the usual formulation of Newtonian gravity, and explore the connection to GR.

2.1. Formulation of the theory and general consequences

Axioms 2.1 (Axioms for Newton–Cartan gravity)**.** Newton–Cartan gravity may be characterised as follows (compare GR!).

(i) Spacetime is a Newtonian manifold (M, τ, h, ∇),

(ii) ideal clocks measure time as defined by τ, and ideal rods measure spatial lengths as defined by the metric ${}^{(n)}h$ induced by h on spacelike vectors,

(iii) free test particles move on timelike geodesics of ∇, and

(iv) the *Newton–Cartan field equation*

$$R_{\mu\nu} = 4\pi G\rho\tau_\mu\tau_\nu \tag{2.1}$$

holds, where $R_{\mu\nu}$ are the components of the Ricci tensor of ∇, G is the gravitational constant, and ρ is the mass density.

Theorem 2.2. *Any* $(3+1)$*-dimensional Galilei manifold with absolute time with a torsion-free Galilei connection satisfying the Newton–Cartan field equation is spatially flat.*

Proof. The field equation implies that the Ricci tensor vanishes on spacelike vectors. According to construction 1.28, this means that all spacelike leaves are Ricci-flat. Since they are 3-dimensional, this implies that they are flat as Riemannian manifolds. □

For analysing timelike geodesics of ∇ (i.e. the worldlines of freely falling test particles), the following result is helpful.

Proposition 2.3. *Let (M, τ, h) be a Galilei manifold with absolute time, and ∇ a Galilei connection on it. Absolute time t as defined by τ (i.e. any local function t such that $\tau = \mathrm{d}t$) is an affine parameter for all timelike geodesics of ∇.*

Proof. Let $\gamma(\lambda)$ be any affinely parametrised geodesic of ∇, i.e. $\nabla_{\gamma'(\lambda)}\gamma'(\lambda) = 0$. This implies $\nabla_{\gamma'(\lambda)}(\tau(\gamma'(\lambda))) = (\nabla_{\gamma'(\lambda)}\tau)(\gamma'(\lambda)) + \tau(\nabla_{\gamma'(\lambda)}\gamma'(\lambda)) = 0$, i.e. that $\tau(\gamma') =: A$ is constant along γ. Therefore, we have

$$t(\gamma(\lambda_f)) - t(\gamma(\lambda_i)) = \int_{\lambda_i}^{\lambda_f} \mathrm{d}\lambda\ \tau(\gamma'(\lambda)) = (\lambda_f - \lambda_i)A, \tag{2.2}$$

i.e. absolute time t is affinely related to λ and therefore an affine parameter itself. □

Notation 2.4. Unless otherwise specified, we will from now on parametrise all timelike curves by absolute time, i.e. such that $\tau(\dot{\gamma}) = 1$.

2.2. Recovering Newtonian gravity

In this section, we will show how to recover the usual coordinate formulation of Newtonian gravity from the geometric framework of Newton–Cartan gravity.

2.2.1. Kinematics of timelike vector fields

In the following, let (M, τ, h) be a Galilei manifold with absolute time and ∇ a torsion-free Galilei connection on it.

Interpretation 2.5. Consider a timelike curve γ that we interpret as the worldline of an observer. A spacelike vector field w along γ which is parallel with respect to ∇, i.e. $\nabla_{\dot{\gamma}} w = 0$, has the interpretation of defining a *non-rotating spacelike vector of constant length* from the point of view of the observer moving along γ. (On exercise sheet 3, we will see why this is a sensible notion of a 'non-rotating' spacelike vector from the observer's point of view.)

Definition 2.6. Let (M, τ, h) be a Galilei manifold with absolute time, and ∇ a torsion-free Galilei connection on it. For any unit timelike vector field v, i.e. $\tau(v) = 1$, we define its *acceleration*

$$\alpha := \nabla_v v \tag{2.3a}$$

(timelike part of ∇v), and we decompose the spacelike part $\nabla^\mu v^\nu$ of ∇v into its antisymmetric part

$$\omega^{\mu\nu} := \nabla^{[\mu} v^{\nu]}, \tag{2.3b}$$

the *twist* of v, its trace

$$\theta := h_{\rho\sigma} \nabla^\rho v^\sigma = \nabla_\rho v^\rho, \tag{2.3c}$$

the *expansion* of v, and its symmetric traceless part

$$\sigma^{\mu\nu} := \nabla^{(\mu} v^{\nu)} - \frac{1}{n} \theta h^{\mu\nu}, \tag{2.3d}$$

the *shear* of v, where $n = \dim M - 1$.

In terms of those, ∇v can be decomposed as

$$\nabla_\mu v^\nu = \tau_\mu \alpha^\nu + \omega_\mu{}^\nu + \sigma_\mu{}^\nu + \frac{1}{n} \theta P^\nu_\mu \,, \tag{2.4}$$

and the Newton–Coriolis form of ∇ with respect to v is

$$\Omega_{\mu\nu} = 2\tau_{[\mu} \alpha_{\nu]} + 2\omega_{\mu\nu} \,. \tag{2.5}$$

Interpretation 2.7. Let v be a unit timelike vector field, and imagine that it describes the flow of a fluid through spacetime, i.e. its flow lines are the worldlines of the fluid particles. A spacelike[1] vector field η satisfying $\mathcal{L}_v \eta = 0$ is transported along the flow of v, i.e. when evaluated along one fixed flow line γ of v, it may be interpreted as the direction from γ to an 'infinitesimally close' flow line over the course of time; see figure 2.1.

Now since ∇ is torsion-free, we have $0 = \mathcal{L}_v \eta = [v, \eta] = \nabla_v \eta - \nabla_\eta v$. This implies

$$\begin{aligned} v^\mu \nabla_\mu \eta^\nu &= \eta^\mu \nabla_\mu v^\nu \\ &\overset{(2.4)}{=} \eta^\mu \left(\omega_\mu{}^\nu + \sigma_\mu{}^\nu + \frac{1}{n} \theta P^\nu_\mu \right) \\ &= -\omega^\nu{}_\mu \eta^\mu + \sigma^\nu{}_\mu \eta^\mu + \frac{1}{n} \theta \eta^\nu. \end{aligned} \tag{2.6a}$$

Evaluated along a fixed flow line γ, this gives

$$\nabla_{\dot\gamma} \eta = -\omega(\eta) + \sigma(\eta) + \frac{1}{n} \theta \eta, \tag{2.6b}$$

[1]Being everywhere spacelike is compatible with $\mathcal{L}_v \eta = 0$: due to Cartan's magic formula, we have $\mathcal{L}_v \tau = \mathrm{d}(\tau(v)) + \mathrm{d}\tau(v, \cdot) = 0$, which implies $\mathcal{L}_v(\tau(\eta)) = \tau(\mathcal{L}_v \eta) = 0$, i.e. $\tau(\eta)$ is constant along the flow of v.

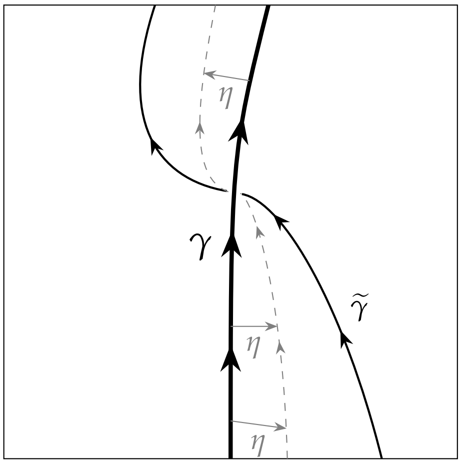

Figure 2.1.: Two flow lines $\gamma, \widetilde{\gamma}$ of a timelike vector field, and the evolution of the connecting vector η to an 'infinitesimally close' flow line along γ

where we treat, at each point, ω and σ as linear maps on the space of spacelike vectors.

This means that the evolution of the 'infinitesimal connecting vector' η along γ, as compared to a non-rotating frame, is at each point in time generated by the linear map $-\omega + \sigma + \frac{1}{n}\theta\,\mathrm{id}$. Since with respect to the metric ${}^{(n)}h$ on space ω is antisymmetric and σ is symmetric traceless, ω generates a rotational motion and σ generates a rotation-free volume-preserving motion[2].

Thus ω gives the rate at which 'infinitesimally close' fluid particles / flow lines rotate around each other, σ describes how their shape is deformed in a volume-preserving way, and θ is the rate at which the volume of an infinitesimal 'ball' of fluid particles increases.

Proposition 2.8. *Let (M, τ, h) be a Galilei manifold with absolute time, ∇ a torsion-free Galilei connection on it, and v any vector field on M. Then the Lie derivative of h in direction of v is given by $(\mathcal{L}_v h)^{\mu\nu} = -2\nabla^{(\mu} v^{\nu)}$.*

Proof. For any one-form α and any vector field X, we have

$$\begin{aligned}(\mathcal{L}_v\alpha)(X) &= \underbrace{v(\alpha(X))}_{=(\nabla_v\alpha)(X)+\alpha(\nabla_v X)} - \alpha(\underbrace{\mathcal{L}_v X}_{=[v,X]=\nabla_v X-\nabla_X v}) \\ &= (\nabla_v\alpha)(X) + \alpha(\nabla_X v).\end{aligned} \tag{2.7}$$

[2] The infinitesimal version of an actual shear in the geometric sense combines this with a rotation.

In index notation, this reads $(\mathcal{L}_v\alpha)_\rho = v^\mu\nabla_\mu\alpha_\rho + \alpha_\mu\nabla_\rho v^\mu$. This implies, for one-forms α, β, that

$$h(\mathcal{L}_v\alpha, \beta) = h(\nabla_v\alpha, \beta) + \alpha_\mu \underbrace{(\nabla_\rho v^\mu)h^{\rho\nu}}_{=\nabla^\nu v^\mu} \beta_\nu \,, \tag{2.8}$$

from which we obtain

$$\begin{aligned}
(\mathcal{L}_v h)(\alpha, \beta) &= \underbrace{v(h(\alpha, \beta))}_{=h(\nabla_v\alpha,\beta)+h(\alpha,\nabla_v\beta)} - h(\mathcal{L}_v\alpha, \beta) - h(\alpha, \mathcal{L}_v\beta) \\
&\overset{(2.8)}{=} -\alpha_\mu(\nabla^\nu v^\mu)\beta_\nu - \beta_\mu(\nabla^\nu v^\mu)\alpha_\nu \\
&= -2\nabla^{(\mu}v^{\nu)}\alpha_\mu\beta_\nu \,.
\end{aligned} \tag{2.9}$$

□

For (unit) timelike v, this matches intuition: $\nabla^{(\mu}v^{\nu)}$ consists of shear and expansion, i.e. encodes (according to interpretation 2.7) the deformation of spatial geometry along the flow of v, which on the other hand is described by the change of the spatial metric ${}^{(n)}h$ along the flow of v (the minus sign comes from inverting h to get ${}^{(n)}h$). Note also that this implies that ∇v is determined by $\mathcal{L}_v h$ and the Newton–Coriolis form Ω.

Definition 2.9. Let (M, τ, h) be a Galilei manifold with absolute time. A timelike vector field v is *rigid* iff $\mathcal{L}_v h = 0$, i.e. iff spatial geometry is constant along its flow.

Proposition 2.10. *Let (M, τ, h) be a Galilei manifold with absolute time, and ∇ a torsion-free Galilei connection on it. For any vector field v with $\nabla^{(\mu}v^{\nu)} = 0$, we have*

$$\nabla^\rho\nabla^\mu v^\nu = -R^{\mu\nu\rho}{}_\sigma v^\sigma. \tag{2.10}$$

Proof. Due to vanishing torsion, we have the Ricci identity

$$2\nabla_{[\mu}\nabla_{\nu]}v^\rho = R^\rho{}_{\sigma\mu\nu}v^\sigma. \tag{2.11}$$

Raising μ, ν and cycling indices, we obtain

$$2\nabla^{[\mu}\nabla^{\nu]}v^\rho = R^\rho{}_\sigma{}^{\mu\nu}v^\sigma, \tag{2.12a}$$

$$2\nabla^{[\nu}\nabla^{\rho]}v^\mu = R^\mu{}_\sigma{}^{\nu\rho}v^\sigma, \tag{2.12b}$$

$$2\nabla^{[\rho}\nabla^{\mu]}v^\nu = R^\nu{}_\sigma{}^{\rho\mu}v^\sigma. \tag{2.12c}$$

Adding the last two equations and subtracting the first, we get

$$\begin{aligned}
-2\nabla^\mu\nabla^{(\nu}v^{\rho)} + 2\nabla^\nu\nabla^{(\rho}v^{\mu)} + 2\nabla^\rho\nabla^{[\mu}v^{\nu]} &= (-R^\rho{}_\sigma{}^{\mu\nu} + R^\mu{}_\sigma{}^{\nu\rho} + R^\nu{}_\sigma{}^{\rho\mu})v^\sigma \\
&= (\cancel{R^{\rho\mu\nu}{}_\sigma} + \cancel{R^{\rho\nu}{}_\sigma{}^\mu} \\
&\quad - R^{\mu\nu\rho}{}_\sigma - \cancel{R^{\mu\rho}{}_\sigma{}^\nu} \\
&\quad - \cancel{R^{\nu\rho\mu}{}_\sigma} - R^{\nu\mu}{}_\sigma{}^\rho)v^\sigma \\
&= -2R^{\mu\nu\rho}{}_\sigma v^\sigma,
\end{aligned} \tag{2.13}$$

where we have used the Bianchi identity and the antisymmetries of the curvature tensor. Due to $\nabla^{(\mu}v^{\nu)} = 0$, we are finished. □

Corollary 2.11. *Let v be a rigid unit timelike vector field on a Galilei manifold with absolute time with a torsion-free Galilei connection ∇. The twist ω of v satisfies*

$$\nabla^\rho\omega^{\mu\nu} = -R^{\mu\nu\rho}{}_\sigma v^\sigma. \tag{2.14}$$

□

2.2.2. Künzle–Ehlers recovery

Here, we will step by step show how to recover a slight generalisation of usual Newtonian gravity from Newton–Cartan gravity. We aim at the formulation of this recovery theorem as given by Malament [Mal12], but we try to provide a more differential-geometric proof.

Let again (M, τ, h) be a Galilei manifold with absolute time, ∇ a torsion-free Galilei connection on it, and v a unit timelike vector field.

Notation 2.12. (i) Let (N, g) be a (pseudo-)Riemannian manifold. For a k-form $\beta \in \Omega^k(N)$, we write

$$(\delta\beta)_{\mu_1\ldots\mu_{k-1}} := -\widetilde{\nabla}^\rho\beta_{\rho\mu_1\ldots\mu_{k-1}}\,, \tag{2.15}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection of g. The operator $\delta\colon \Omega^k(N) \to \Omega^{k-1}(N)$ is called the *codifferential*[3].

[3] δ is formally adjoint to the exterior derivative d, i.e. satisfies

$$\frac{1}{k!}\int_N \mathrm{vol}_g\, \alpha_{\mu_1\ldots\mu_k}(\mathrm{d}\beta)^{\mu_1\ldots\mu_k} = \frac{1}{(k-1)!}\int_N \mathrm{vol}_g\,(\delta\alpha)_{\mu_1\ldots\mu_{k-1}}\beta^{\mu_1\ldots\mu_{k-1}} \tag{2.16}$$

for any k-form α and $(k-1)$-form β with compact support.

The codifferential may also be expressed in terms of the exterior derivative d and the Hodge star operator with respect to g, but we will use only (2.15) instead, i.e. 'codifferential is minus covariant divergence'. This also saves us from a lot of trouble regarding sign conventions for the Hodge operator.

(ii) For any spatial leaf Σ of our Galilei manifold (M, τ, h) with absolute time, we will denote by ${}^{(n)}\delta$ the codifferential of the Riemannian manifold $(\Sigma, {}^{(n)}h)$.

Proposition 2.13. *Consider a k-form β and a $(k-1)$-form κ on M that are purely spacelike with respect to v.*

(i) The equation

$$-\nabla^\rho \beta_{\rho\mu_1\dots\mu_{k-1}} = \kappa_{\mu_1\dots\mu_{k-1}} \tag{2.17}$$

is equivalent to

$$ {}^{(n)}\delta(\beta|_\Sigma) = \kappa|_\Sigma \text{ on all spatial leaves } \Sigma. \tag{2.18}$$

(ii) The equation

$$k\nabla^{[\mu_1}\kappa^{\mu_2\dots\mu_k]} = \beta^{\mu_1\dots\mu_k} \tag{2.19}$$

is equivalent to

$$\mathrm{d}(\kappa|_\Sigma) = \beta|_\Sigma \text{ on all spatial leaves } \Sigma. \tag{2.20}$$

Proof. (i) β is purely spacelike, such that we have $\nabla^\rho \beta_{\rho\mu_1\dots\mu_{k-1}} = \overset{(n)}{\nabla}{}^\rho \beta_{\rho\mu_1\dots\mu_{k-1}} = h^{\rho\sigma}\overset{(n)}{\nabla}_\sigma \beta_{\rho\mu_1\dots\mu_{k-1}}$. Restricted to any spatial leaf Σ, $\overset{(n)}{\nabla}$ is the Levi-Civita connection of $(\Sigma, {}^{(n)}h)$, and thus restriction of (2.17) to Σ gives (2.18).

(ii) Since κ is purely spacelike, we have $k\nabla^{[\mu_1}\kappa^{\mu_2\dots\mu_k]} = k\overset{(n)}{\nabla}{}^{[\mu_1}\kappa^{\mu_2\dots\mu_k]}$. Restricted to Σ, these are the components of $\mathrm{d}(\kappa|_\Sigma)$, expressed in terms of the Levi-Civita connection $\overset{(n)}{\nabla}$ of $(\Sigma, {}^{(n)}h)$, and thus restriction of (2.19) to Σ gives (2.20). ☐

In the following, we view the twist ω of v as a two-form and the acceleration α of v as a one-form, both of which are purely spacelike with respect to v.

Lemma 2.14. *If ∇ is Newtonian, then on any spatial leaf Σ we have $\mathrm{d}(\omega|_\Sigma) = 0$.*

Proof. According to (2.5), the twist is one half of the spatial part of the Newton–Coriolis form, i.e. $\omega|_\Sigma = \frac{1}{2}\,\Omega|_\Sigma$. Now ∇ being Newtonian means $\mathrm{d}\Omega = 0$ (theorem 1.27), which implies $\mathrm{d}(\omega|_\Sigma) = \frac{1}{2}\mathrm{d}(\Omega|_\Sigma) = \frac{1}{2}\,(\mathrm{d}\Omega)|_\Sigma = 0$, since d commutes with pullback and therefore with restriction to submanifolds. ☐

Lemma 2.15. *If ∇ satisfies the Newton–Cartan field equation and v is rigid, then on any spatial leaf Σ we have ${}^{(n)}\delta(\omega|_\Sigma) = 0$.*

Proof. By corollary 2.11 and the field equation (2.1), we have

$$\begin{aligned} h_{\rho\mu}\nabla^{\rho}\omega^{\mu\nu} &= -h_{\rho\mu}R^{\mu\nu\rho}{}_{\sigma}v^{\sigma} \\ &= -R^{\nu}{}_{\sigma}v^{\sigma} \\ &\overset{(2.1)}{=} 0, \end{aligned} \tag{2.21}$$

which by proposition 2.13 (i) may be invariantly written as ${}^{(n)}\delta(\omega|_{\Sigma}) = 0$. □

Lemma 2.16. *If ∇ is Newtonian and v is rigid, then on any spatial leaf Σ acceleration and twist satisfy* $\mathrm{d}(\alpha|_{\Sigma}) = 2(\nabla_{v}\omega)|_{\Sigma}$.

Proof. According to (2.5), we have $\Omega = \tau \wedge \alpha + 2\omega$; so ∇ being Newtonian means (theorem 1.27) that $0 = \mathrm{d}\Omega = -\tau \wedge \mathrm{d}\alpha + 2\mathrm{d}\omega$. In components, this becomes

$$\begin{aligned} 0 &= -3\tau_{[\mu}(\mathrm{d}\alpha)_{\nu\rho]} + 2\cdot 3\partial_{[\mu}\omega_{\nu\rho]} \\ &= -6\tau_{[\mu}\partial_{\nu}\alpha_{\rho]} + 6\partial_{[\mu}\omega_{\nu\rho]}\ . \end{aligned} \tag{2.22}$$

Due to ∇ being torsion-free, we may replace the anti-symmetrised partial derivatives by anti-symmetrised covariant derivatives with respect to ∇, obtaining

$$\begin{aligned} 0 &= -6\tau_{[\mu}\nabla_{\nu}\alpha_{\rho]} + 6\nabla_{[\mu}\omega_{\nu\rho]} \\ &= -2\tau_{\mu}\nabla_{[\nu}\alpha_{\rho]} - 2\tau_{\nu}\nabla_{[\rho}\alpha_{\mu]} - 2\tau_{\rho}\nabla_{[\mu}\alpha_{\nu]} \\ &\quad + 2\nabla_{\mu}\omega_{\nu\rho} + 2\nabla_{\nu}\omega_{\rho\mu} + 2\nabla_{\rho}\omega_{\mu\nu}\ . \end{aligned} \tag{2.23}$$

Then raising ν, ρ and contracting with v^{μ}, we obtain

$$0 = -2\nabla^{[\nu}\alpha^{\rho]} + 2v^{\mu}\nabla_{\mu}\omega^{\nu\rho} + 2v^{\mu}\nabla^{\nu}\omega^{\rho}{}_{\mu} + 2v^{\mu}\nabla^{\rho}\omega_{\mu}{}^{\nu}. \tag{2.24}$$

Due to ω being purely spacelike and v being rigid, we may rewrite the last two terms of (2.24) as

$$\begin{aligned} 2v^{\mu}\nabla^{\nu}\omega^{\rho}{}_{\mu} + 2v^{\mu}\nabla^{\rho}\omega_{\mu}{}^{\nu} &= -2\omega^{\rho}{}_{\mu}\underbrace{\nabla^{\nu}v^{\mu}}_{=\omega^{\nu\mu}} - 2\omega_{\mu}{}^{\nu}\underbrace{\nabla^{\rho}v^{\mu}}_{=\omega^{\rho\mu}} \\ &= -2\omega^{\rho}{}_{\mu}\omega^{\nu\mu} + 2\omega^{\nu}{}_{\mu}\omega^{\rho\mu} \\ &= 0. \end{aligned} \tag{2.25}$$

Thus (2.24) becomes

$$2v^{\mu}\nabla_{\mu}\omega^{\nu\rho} = 2\nabla^{[\nu}\alpha^{\rho]}\ , \tag{2.26}$$

which by proposition 2.13 (ii) may be written as $\mathrm{d}(\alpha|_{\Sigma}) = 2(\nabla_{v}\omega)|_{\Sigma}$. □

Now, in addition to ∇, consider the unique torsion-free Galilei connection $\overset{v}{\nabla}$ that has vanishing Newton–Coriolis form with respect to v. According to the classification theorem 1.16, we have

$$\Gamma^{\rho}_{\mu\nu} = \overset{v}{\Gamma}{}^{\rho}_{\mu\nu} + \tau_{(\mu}\Omega_{\nu)}{}^{\rho}. \tag{2.27}$$

Corollary 2.17. *If ∇ is Newtonian and v is rigid, we have $\mathrm{d}(\alpha|_\Sigma) = 2(\overset{v}{\nabla}_v\,\omega)|_\Sigma$.*

Proof. A direct calculation using (2.27) and (2.5) shows that $\nabla_v\,\omega = \overset{v}{\nabla}_v\,\omega$, so the result follows from lemma 2.16. □

Lemma 2.18. *If v is rigid, then $\overset{v}{\nabla}v = 0$.*

Proof. Since v is rigid, we have $\overset{v}{\nabla}{}^{(\mu}v^{\nu)} = 0$; since the Newton–Coriolis form of $\overset{v}{\nabla}$ with respect to v vanishes, we also have $\overset{v}{\nabla}{}^{[\mu}v^{\nu]} = 0$ and $\overset{v}{\nabla}_v v = 0$, i.e. $\overset{v}{\nabla}v = 0$. □

Lemma 2.19. *Let v be rigid, and (M,τ,h) spatially flat. Then $\overset{v}{\nabla}$ is flat.*

Proof. Fix any spatial leaf Σ. Since it is flat as a Riemannian manifold (spatial flatness), there exists (locally) a parallel frame of vector fields on Σ, i.e. vector fields $\tilde{\mathrm{e}}_a \in \Gamma(T\Sigma)$, $a \in \{1,\ldots,n\}$ which satisfy $\overset{(n)}{\nabla}_{\tilde{\mathrm{e}}_a}\tilde{\mathrm{e}}_b = 0$ and pointwise are a basis of $T_p\Sigma$.

We extend these vector fields along the flow of v to obtain vector fields e_a defined on a neighbourhood of Σ in M, i.e.

$$\mathrm{e}_a|_\Sigma = \tilde{\mathrm{e}}_a\,, \quad 0 = \mathcal{L}_v\mathrm{e}_a = [v,\mathrm{e}_a]. \tag{2.28}$$

Due to vanishing torsion and $\overset{v}{\nabla}v = 0$ (lemma 2.18), we have

$$0 = [v,\mathrm{e}_a] = \overset{v}{\nabla}_v\mathrm{e}_a - \underbrace{\overset{v}{\nabla}_{\mathrm{e}_a}v}_{=0} = \overset{v}{\nabla}_v\mathrm{e}_a\,. \tag{2.29}$$

Furthermore, since on each spatial leaf, $\overset{(n)}{\nabla}$ is the Levi-Civita connection of the induced spatial metric, $\overset{(n)}{\nabla}_{\mathrm{e}_a}\mathrm{e}_b$ can be expressed purely in terms of h, e_a and e_b. Due to $\mathcal{L}_v h = 0$, $\mathcal{L}_v\mathrm{e}_a = 0$ and $\mathcal{L}_v\mathrm{e}_b = 0$, we thus have $\mathcal{L}_v(\overset{(n)}{\nabla}_{\mathrm{e}_a}\mathrm{e}_b) = 0$. Together with $(\overset{(n)}{\nabla}_{\mathrm{e}_a}\mathrm{e}_b)|_\Sigma = (\overset{(n)}{\nabla}_{\tilde{\mathrm{e}}_a}\tilde{\mathrm{e}}_b) = 0$, this implies

$$0 = \overset{(n)}{\nabla}_{\mathrm{e}_a}\mathrm{e}_b = \overset{v}{\nabla}_{\mathrm{e}_a}\mathrm{e}_b\,. \tag{2.30}$$

Combined, we thus have a local frame $\{v,\mathrm{e}_a\}$ that satisfies

$$\overset{v}{\nabla}v = 0, \quad \overset{v}{\nabla}\mathrm{e}_a = 0, \tag{2.31}$$

i.e. that is parallel with respect to $\overset{v}{\nabla}$, which means that $\overset{v}{\nabla}$ is flat. □

Lemma 2.20. *If ∇ satisfies the Newton–Cartan field equation, v is rigid and (M, τ, h) is spatially flat, then on any spatial leaf Σ we have*

$$-{}^{(n)}\delta(\alpha|_\Sigma) = 4\pi G\rho - \omega_{\mu\nu}\omega^{\mu\nu}, \tag{2.32}$$

where ρ is the mass density.

Proof. Writing $\Gamma^\rho_{\mu\nu} = \overset{v}{\Gamma}{}^\rho_{\mu\nu} + S^\rho{}_{\mu\nu}$, the curvature tensor of ∇ can be expressed in terms of the curvature tensor $\overset{v}{R}$ of $\overset{v}{\nabla}$ as

$$R^\mu{}_{\nu\rho\sigma} = \overset{v}{R}{}^\mu{}_{\nu\rho\sigma} + \overset{v}{\nabla}_\rho S^\mu{}_{\sigma\nu} - \overset{v}{\nabla}_\sigma S^\mu{}_{\rho\nu} + S^\mu{}_{\rho\lambda} S^\lambda{}_{\sigma\nu} - S^\mu{}_{\sigma\lambda} S^\lambda{}_{\rho\nu}\,, \tag{2.33}$$

so due to flatness of $\overset{v}{\nabla}$ (lemma 2.19) we have

$$\begin{aligned} v^\nu v^\sigma R_{\nu\sigma} &= v^\nu v^\sigma R^\mu{}_{\nu\mu\sigma} \\ &= v^\nu v^\sigma (\overset{v}{\nabla}_\mu S^\mu{}_{\sigma\nu} - \overset{v}{\nabla}_\sigma S^\mu{}_{\mu\nu} + S^\mu{}_{\mu\lambda} S^\lambda{}_{\sigma\nu} - S^\mu{}_{\sigma\lambda} S^\lambda{}_{\mu\nu}). \end{aligned} \tag{2.34}$$

Equation (2.27) together with $\Omega = \tau \wedge \alpha + 2\omega$ (equation (2.5)) gives

$$S^\rho{}_{\mu\nu} = \tau_{(\mu}\Omega_{\nu)}{}^\rho = \tau_\mu\tau_\nu\alpha^\rho + 2\tau_{(\mu}\omega_{\nu)}{}^\rho, \tag{2.35}$$

and in particular $S^\mu{}_{\mu\nu} = 0$. Using these, (2.34) becomes

$$\begin{aligned} v^\nu v^\sigma R_{\nu\sigma} &= v^\nu v^\sigma (\tau_\sigma\tau_\nu \overset{v}{\nabla}_\mu \alpha^\mu + 2\tau_{(\sigma}\overset{v}{\nabla}_{|\mu|}\omega_{\nu)}{}^\mu - (\tau_\sigma\tau_\lambda\alpha^\mu + 2\tau_{(\sigma}\omega_{\lambda)}{}^\mu)(\tau_\mu\tau_\nu\alpha^\lambda + 2\tau_{(\mu}\omega_{\nu)}{}^\lambda)) \\ &= \overset{v}{\nabla}_\mu\alpha^\mu + 2v^\nu\overset{v}{\nabla}_\mu\omega_\nu{}^\mu - \omega_\lambda{}^\mu\omega_\mu{}^\lambda, \end{aligned} \tag{2.36}$$

where we have used that α and ω are purely spacelike. Due to $\overset{v}{\nabla} v = 0$ (lemma 2.18), we have $v^\nu \overset{v}{\nabla}_\mu \omega_\nu{}^\mu = \overset{v}{\nabla}_\mu (v^\nu\omega_\nu{}^\mu) = 0$; and due to the field equation (2.1), we have $v^\nu v^\sigma R_{\nu\sigma} = 4\pi G\rho$. Thus (2.36) is equivalent to

$$4\pi G\rho = \overset{v}{\nabla}_\mu\alpha^\mu + \omega_{\mu\nu}\omega^{\mu\nu}, \tag{2.37}$$

finishing the proof (with proposition 2.13 (i)). □

Combining everything, we arrive at the final result.

Theorem 2.21 (Künzle–Ehlers recovery theorem)**.** *Let (M, τ, h, ∇) be a Newtonian manifold that satisfies the Newton–Cartan field equation and is spatially flat. Let v be a rigid unit timelike vector field on (M, τ, h), let α and ω be its acceleration and twist with respect to ∇, and denote by $\overset{v}{\nabla}$ the unique torsion-free Galilei connection that has vanishing Newton–Coriolis form with respect to v. Then* *(i)* $\overset{v}{\nabla} v = 0$,

(ii) $\overset{v}{\nabla}$ *is flat,*

(iii) a timelike curve γ *parametrised by absolute time is a geodesic of* ∇ *iff*

$$(\overset{v}{\nabla}_{\dot{\gamma}} \dot{\gamma})^\nu = -\alpha^\nu + 2\omega^\nu{}_\mu \dot{\gamma}^\mu \tag{2.38}$$

along γ*, and*

(iv) on any spatial leaf Σ*, we have the 'recovered field equations'*

$$\mathrm{d}(\omega|_\Sigma) = 0, \tag{2.39a}$$
$${}^{(n)}\delta(\omega|_\Sigma) = 0, \tag{2.39b}$$
$$\mathrm{d}(\alpha|_\Sigma) = 2(\overset{v}{\nabla}_v\, \omega)|_\Sigma\,, \tag{2.39c}$$
$$-{}^{(n)}\delta(\alpha|_\Sigma) = 4\pi G\rho - \omega_{\mu\nu}\omega^{\mu\nu}, \tag{2.39d}$$

where ρ *is the mass density.*

Proof. This is a combination of lemmas 2.14, 2.15, 2.18, 2.19, 2.20 and corollary 2.17. The only remaining statement that we have to prove is (2.38), i.e. the equation of motion for test particles with respect to $\overset{v}{\nabla}$, which follows easily from (2.27) and (2.5): for any curve γ with $\tau(\dot{\gamma}) = 1$, these give

$$\begin{aligned}(\nabla_{\dot{\gamma}}\dot{\gamma})^\nu &= (\overset{v}{\nabla}_{\dot{\gamma}}\dot{\gamma})^\nu + \tau_{(\mu}\Omega_{\lambda)}{}^\nu \dot{\gamma}^\mu\dot{\gamma}^\lambda \\ &= (\overset{v}{\nabla}_{\dot{\gamma}}\dot{\gamma})^\nu + (\tau_\mu\tau_\lambda\alpha^\nu + 2\tau_{(\mu}\omega_{\lambda)}{}^\nu)\dot{\gamma}^\mu\dot{\gamma}^\lambda \\ &= (\overset{v}{\nabla}_{\dot{\gamma}}\dot{\gamma})^\nu + \alpha^\nu - 2\omega^\nu{}_\lambda\dot{\gamma}^\lambda. \end{aligned} \tag{2.40}$$

□

Construction 2.22 (Recovery in adapted coordinates)**.** Suppose that our Galilei manifold (M, τ, h) is spatially flat. On one spatial leaf Σ, we introduce orthonormal coordinates $(\tilde{x}^a)$, $a \in \{1, \ldots, n\}$, i.e. coordinates in which the metric takes the form ${}^{(n)}h_{ab} = \delta_{ab}$. Now using our unit timelike vector field v, we can extend the coordinates to a neighbourhood of Σ in M: starting at a point on Σ with coordinates (x^a) and following the flow of v for time parameter t, the resulting point in M gets coordinates (t, x^a). Put differently, the coordinates are defined such that the flow lines of v are lines of constant spatial coordinates (x^a) and varying t, i.e.

$$v = \frac{\partial}{\partial t} \text{ with respect to } (t, x^a); \tag{2.41}$$

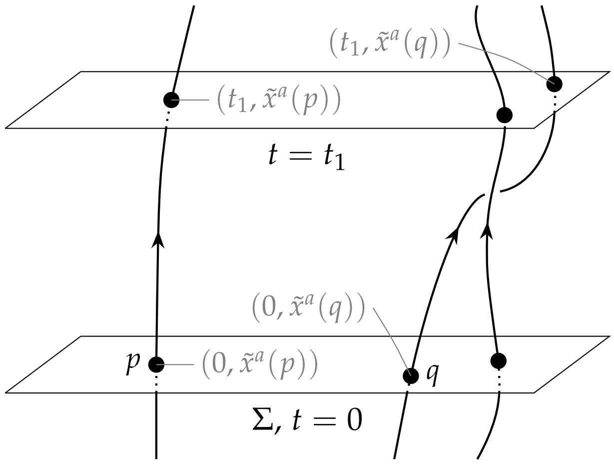

Figure 2.2.: The definition of coordinates adapted to a timelike vector field

and the (x^a) can be imagined to be realised by coordinate labels that are mounted to particles flowing with v, see figure 2.2.

Now by construction, when evaluated on the original spatial leaf Σ, the coordinate vector fields $\mathrm{e}_a := \frac{\partial}{\partial x^a}$ agree with the fields $\tilde{\mathrm{e}}_a := \frac{\partial}{\partial \tilde{x}^a}$; and since they are coordinate fields, the e_a all commute with $v = \frac{\partial}{\partial t}$, i.e. are transported along the flow of v. Because the original coordinates $\tilde{x}^a$ are orthonormal, the $\tilde{\mathrm{e}}_a$ are parallel with respect to the Levi-Civita connection $\overset{(n)}{\nabla}$ of $(\Sigma, {}^{(n)}h)$. So we have

$$\mathrm{e}_a|_\Sigma = \tilde{\mathrm{e}}_a \,, \quad \mathcal{L}_v \mathrm{e}_a = 0, \quad \overset{(n)}{\nabla}_{\tilde{\mathrm{e}}_a} \tilde{\mathrm{e}}_b = 0 \text{ on } \Sigma. \tag{2.42}$$

Thus, if v is rigid, we know from the proof of flatness of $\overset{v}{\nabla}$ (lemma 2.19) that the frame $\{v, \mathrm{e}_a\}$ is parallel with respect to $\overset{v}{\nabla}$, i.e. that

$$\overset{v}{\Gamma}{}^\rho_{\mu\nu} = 0 \text{ in the coordinates } (t, x^a). \tag{2.43a}$$

In particular, the x^a are flat coordinates on *all* spatial leaves. Since $\mathcal{L}_v h = 0$, they are in fact orthonormal, and we have

$$\tau = \mathrm{d}t, \quad h^{tt} = h^{ta} = 0, \; h^{ab} = \delta^{ab} \text{ in the coordinates } (t, x^a). \tag{2.43b}$$

Expressed in those adapted coordinates, the test particle equation of motion (2.38) therefore takes the form

$$\ddot{\gamma}^a = -\alpha^a + 2\omega^a{}_b \dot{\gamma}^b, \tag{2.44}$$

i.e. in those coordinates $-\alpha$ plays the role of the Newtonian gravitational field (gravitational acceleration), while the local rate $-\omega$ of rotation of the spatial frame (e_a) gives rise to a Coriolis-force term (in the three-dimensional case, writing $-\omega_{ab} =: \varepsilon_{abc}\omega^c$, the term becomes the familiar $2\omega^a{}_b\dot{\gamma}^b = -2(\vec{\omega}\times\dot{\vec{\gamma}})^a$). This is the reason for $\Omega = \tau\wedge\alpha + 2\omega$ being called the Newton–Coriolis form.

The recovered field equations (2.39) become

$$\partial_{[a}\omega_{bc]} = 0, \quad \partial_a\omega^{ab} = 0, \tag{2.45a}$$

$$\partial_{[a}\alpha_{b]} = \partial_t\omega_{ab}, \quad \partial_a\alpha^a = 4\pi G\rho - \omega_{ab}\omega^{ab}, \tag{2.45b}$$

i.e. on each spatial leaf the twist field ω has to satisfy the constraints (2.45a), and by (2.45b) mass density and twist together determine the gravitational field $-\alpha$ (up to homogeneous solutions of (2.45b), i.e. harmonic one-forms).

We thus have recovered the usual formulation of Newtonian gravity (in n-dimensional space) in Cartesian coordinates with respect to potentially rotating reference frames.

The formulation of the recovery theorem in terms of the flat connection $\overset{v}{\nabla}$ may be seen as a coordinate-free method to state what the equations look like in coordinates adapted to the 'observer vector field' v.

Remark 2.23. The converse of the recovery theorem is true as well. Suppose we are given a Galilei manifold (M, τ, h) with absolute time that is spatially flat, and a rigid unit timelike vector field v on it. We know that the associated torsion-free Galilei connection $\overset{v}{\nabla}$ with $\overset{v}{\nabla}v = 0$ is flat.

Now, given any purely spacelike one-form α and purely spacelike two-form ω, we can define the torsion-free Galilei connection ∇ that has Newton–Coriolis form $\Omega := \tau\wedge\alpha + 2\omega$ with respect to v, such that α and ω are then the acceleration and twist of v with respect to ∇. By essentially going backwards through the arguments of this section, one can show that if α and ω satisfy the equations (2.39), the resulting ∇ will be Newtonian (i.e. $\mathrm{d}\Omega = 0$) and satisfy the Newton–Cartan field equation.

We can even start one step further back: Given a solution of standard Newtonian gravity, i.e. a solution α, ω to the equations (2.45) on (some portion of) $M = \mathbb{R}^{n+1}$ with coordinates (t, x^a), we can define $\tau = \mathrm{d}t$, $h := \delta^{ab}\partial_a\otimes\partial_b$ and $v := \partial_t$ and then continue as above to construct a spacetime satisfying the axioms for Newton–Cartan gravity.

This procedure of obtaining a Newton–Cartan spacetime from Newtonian gravity is sometimes called *geometrisation*.

Note that the recovery theorem depends crucially on the rigidity of the 'observer vector field' v with respect to which we have performed the recovery (i.e. which we use, in the coordinate formulation of construction 2.22, to identify the spatial coordinates on different spatial leaves). If there were no rigid vector fields, we would not be able to recover Newtonian gravity. But fortunately spatial flatness ensures that we may construct rigid vector fields:

Proposition 2.24. *Let (M, τ, h) be a Galilei manifold with absolute time that is spatially flat. Then for any timelike worldline γ one can construct a rigid timelike vector field v (in a neighbourhood of γ) that along γ agrees with $\dot{\gamma}$.*

Proof. Since the spatial leaves are flat, on each of them we can introduce orthonormal coordinates for the induced metric. In particular, we can introduce coordinates (t, x^a) in a neighbourhood of γ in M such that t is absolute time, the x^a are orthonormal coordinates on each spatial leaf, and $x^a(\gamma(t)) = 0$ for all t.[4] The x^a being orthonormal with respect to the induced metric ${}^{(n)}h$ means that in the coordinates (t, x^a), we have ${}^{(n)}h_{ab} = \delta_{ab}$. Together with $\tau = \mathrm{d}t$ this implies that

$$h^{ab} = \delta^{ab} \text{ in the coordinates } (t, x^a). \tag{2.46}$$

Now we define

$$v := \frac{\partial}{\partial t} \text{ in the coordinates } (t, x^a), \tag{2.47}$$

i.e. v points in the direction of constant spatial coordinates x^a. By construction, v agrees with $\dot{\gamma}$ along γ.

Furthermore, since the Lie derivative commutes with the exterior derivative, we have $\mathcal{L}_v(\mathrm{d}x^a) = \mathrm{d}(\mathcal{L}_v x^a) = \mathrm{d}(\mathrm{d}x^a(v)) = \mathrm{d}(v^a) = \mathrm{d}(0) = 0$, and similarly $\mathcal{L}_v(\mathrm{d}t) = \mathrm{d}(v^t) = \mathrm{d}(1) = 0$. By the Leibniz rule for the Lie derivative, we obtain

$$\begin{aligned}
(\mathcal{L}_v h)^{ab} &= (\mathcal{L}_v h)(\mathrm{d}x^a, \mathrm{d}x^b) \\
&= v(h^{ab}) - h(\mathcal{L}_v(\mathrm{d}x^a), \mathrm{d}x^b) - h(\mathrm{d}x^a, \mathcal{L}_v(\mathrm{d}x^b)) \\
&= v(\delta^{ab}) = 0
\end{aligned} \tag{2.48a}$$

and, for any one-form β,

$$\begin{aligned}
(\mathcal{L}_v h)(\mathrm{d}t, \beta) &= v(h(\mathrm{d}t, \beta)) - h(\mathcal{L}_v(\mathrm{d}t), \beta) - h(\mathrm{d}t, \mathcal{L}_v \beta) \\
&= v(h(\tau, \beta)) - h(\tau, \mathcal{L}_v \beta) \\
&= 0.
\end{aligned} \tag{2.48b}$$

This shows $\mathcal{L}_v h = 0$, i.e. rigidity of v. □

[4] Explicitly, such coordinates may be constructed by taking a smooth orthonormal frame of spacelike vector fields e_a defined along γ, and on each leaf Σ at time t introducing normal coordinates around $\gamma(t)$ with respect to the basis $\{\mathrm{e}^a(t)\}$ of $T_{\gamma(t)}\Sigma$.
In the following construction of v, the e_a become the 'connecting vectors' from γ to 'infinitesimally close' flow lines of v, so by choosing the e_a to be parallelly transported along γ, we could arrange for the twist of v to vanish along γ.

2.2.3. Trautman's condition of absolute rotation

The appearance of the twist field ω in the Künzle–Ehlers recovery theorem 2.21 might seem a little strange: we would arrive at the standard formulation of Newtonian gravity in inertial reference frames in the case $\omega = 0$, i.e. for *twist-free* observer vector fields v. However, if we had a twist-free rigid field v, by corollary 2.11 this would imply $R^{\mu\nu\rho}{}_{\sigma}v^{\sigma} = -\nabla^{\rho}\omega^{\mu\nu} = 0$, which in general need not be true. Therefore, in the general case of Newton–Cartan gravity, *twist-free rigid vector fields need not exist*, due to a 'curvature obstruction'.

In the formulation of Newton–Cartan gravity by Trautman [Tra63], an additional curvature condition is demanded which, as we will see, guarantees the existence of twist-free rigid vector fields.

Definition 2.25. A spatially flat Galilei manifold with absolute time with a torsion-free Galilei connection is said to have *absolute rotation* iff the connection's curvature tensor satisfies

$$R^{\mu\nu}{}_{\rho\sigma} = 0. \tag{2.49}$$

Proposition 2.26. *Consider a spatially flat Galilei manifold with absolute time with a torsion-free Galilei connection. The following statements are equivalent:*

(i) The manifold has absolute rotation.

(ii) For all timelike vectors ξ, we have

$$R^{\mu\nu\rho}{}_{\sigma}\xi^{\sigma} = 0. \tag{2.50}$$

(iii) Given any rigid unit timelike vector field, its twist ω is spatially constant, i.e.

$$\nabla^{\rho}\omega^{\mu\nu} = 0. \tag{2.51}$$

Proof. Equivalence of (i) and (ii) follows from spatial flatness and antisymmetry of the curvature tensor in its last two indices.

Given (ii), we directly obtain (iii) by corollary 2.11. Conversely, given any timelike vector $\xi \in T_pM$, which without loss of generality we may assume unit timelike, we can extend it to some timelike curve γ in a neighbourhood of p, which by proposition 2.24 can in turn be extended to a rigid vector field v, which then satisfies $v|_p = \xi$. By corollary 2.11, we then have $R^{\mu\nu\rho}{}_{\sigma}\xi^{\sigma} = (R^{\mu\nu\rho}{}_{\sigma}v^{\sigma})|_p = -(\nabla^{\rho}\omega^{\mu\nu})|_p$, and thus (iii) implies (ii). □

The preceding proposition is the reason for the name 'absolute rotation'[5]: the twist of a rigid vector field v being spatially constant means that in all of space, the flow lines of v rotate around each other with the same rate of rotation / angular velocity.

Remark 2.27. In the spatially flat case, we have $R^{\mu\nu\rho}{}_{\sigma}(v^\sigma - \tilde{v}^\sigma) = 0$ for any two unit timelike vector fields v and $\tilde{v}$. For rigid fields, by corollary 2.11 this implies that

$$\nabla^\rho \omega^{\mu\nu} = \nabla^\rho \tilde{\omega}^{\mu\nu}, \tag{2.52}$$

i.e. the spatial derivative of the twist is the same for *all* rigid unit timelike vector fields. In particular, the twist is spatially constant either for all rigid fields (absolute rotation) or for none.

In particular, absolute rotation is already implied by just *one* rigid unit timelike vector field having spatially constant twist.

Proposition 2.28. *Let (M, τ, h, ∇) be a spatially flat Galilei manifold with absolute time with a torsion-free Galilei connection that has absolute rotation. Let v be a rigid unit timelike vector field, and γ one fixed flow line of v. Then there is a unique twist-free rigid timelike field $\tilde{v}$ that along γ agrees with v.*

Proof. The idea of the proof is that due to v 'rotating everywhere with the same speed', we may 'counter-rotate'.

Due to rigidity, we have $\nabla^\mu v^\nu = \omega^{\mu\nu}$, and $\tilde{v}$ being rigid and twist-free would mean $\nabla^\mu \tilde{v}^\nu = 0$. This is true if and only if the spacelike difference vector field $X := \tilde{v} - v$ satisfies

$$\overset{(n)}{\nabla}{}^\mu X^\nu = -\omega^{\mu\nu}, \; X|_\gamma = 0. \tag{2.53}$$

This is an independent equation for X on each spatial leaf Σ. In orthogonal coordinates (x^a) on Σ such that $x^a(\gamma(t)) = 0$ for the corresponding value of t, this equation becomes

$$\partial_a X^b = \omega^b{}_a, \; X^a(0) = 0, \tag{2.54}$$

where the components $\omega^b{}_a$ are constant ($\omega|_\Sigma$ is covariantly constant, i.e. its components in orthogonal coordinates are constant). This system has the unique solution

$$X^a(x) = \omega^a{}_b x^b. \tag{2.55}$$

□

[5] The interpretation of the curvature condition (2.49) in terms of rotation was, as far as I (the lecturer) know, not given by Trautman, at least not explicitly. To my knowledge, it first appears in explicit form in Ehlers' 1981 article on frame theory [Ehl81].

Theorem 2.29 (Trautman recovery theorem). *Let (M, τ, h, ∇) be a Newtonian manifold that satisfies the Newton–Cartan field equation, is spatially flat and has absolute rotation. Let v be a twist-free rigid unit timelike vector field on (M, τ, h, ∇), and denote by $\overset{v}{\nabla}$ the unique torsion-free Galilei connection that has vanishing Newton–Coriolis form with respect to v. Then locally there exists a function ϕ on M, unique up to addition of a constant on each spatial leaf, such that*

(i) a timelike curve γ parametrised by absolute time is a geodesic of ∇ iff

$$(\overset{v}{\nabla}_{\dot\gamma}\dot\gamma)^\nu = -\overset{v}{\nabla}{}^\nu\phi \tag{2.56}$$

along γ, and

(ii) ϕ satisfies Poisson's equation

$$\overset{v}{\nabla}_\mu\overset{v}{\nabla}{}^\mu\phi = 4\pi G\rho, \tag{2.57}$$

where ρ is the mass density.

Proof. We apply the Künzle–Ehlers recovery theorem 2.21 in the case $\omega = 0$. The recovered field equations (2.39) reduce to

$$\mathrm{d}(\alpha|_\Sigma) = 0, \tag{2.58a}$$
$$-{}^{(n)}\delta(\alpha|_\Sigma) = 4\pi G\rho. \tag{2.58b}$$

The first of these tells us that on each spatial leaf Σ, $\alpha|_\Sigma$ is closed, such that by the Poincaré lemma we locally have $\alpha|_\Sigma = \mathrm{d}(\phi_\Sigma)$ for some function ϕ_Σ on Σ. Combining those for different Σ, we obtain a (locally defined) function ϕ on M which satisfies[6]

$$\alpha|_\Sigma = \mathrm{d}(\phi|_\Sigma). \tag{2.59}$$

Inserting this into (2.58b), we obtain Poisson's equation; and inserting it into the test particle equation of motion (2.38) from the Künzle–Ehlers recovery theorem, we obtain the equation of motion (2.56). □

Remark 2.30. In coordinates adapted to v as in construction 2.22, this is the standard formulation of Newtonian gravity with gravitational potential ϕ: The test particle equation of motion becomes

$$\ddot\gamma^a = -\delta^{ab}\partial_b\phi, \tag{2.60}$$

and the field equation for the potential ϕ becomes

$$\delta^{ab}\partial_a\partial_b\phi = 4\pi G\rho. \tag{2.61}$$

[6]Note that (2.59) means that as a form on spacetime, we have $\alpha = \mathrm{d}\phi + f\tau$ for some function f, which by $\alpha_\mu v^\mu = 0$ is fixed to be $f = -v(\phi)$.

Remark 2.31. The possibility of motions that are rigid but whose angular velocity varies throughout space might seem unintuitive, and so one might be tempted to discard Newton–Cartan spacetimes not satisfying the absolute rotation condition as 'unphysical' – after all, in usual Newtonian mechanics, rotation *is* absolute. However, when considering Newton–Cartan gravity as a limit of GR (see section 2.3), the absolute rotation condition is not automatically satisfied; i.e. there are general-relativistic spacetimes with 'Newtonian' limits in which rotation is *not* absolute.

As was shown by Ehlers (suggested by Künzle), the absolute rotation condition follows from a weak notion of asymptotic flatness:

Proposition 2.32. *Let (M, τ, h, ∇) be a Newtonian manifold that satisfies the Newton–Cartan field equation and is spatially flat, and whose spatial leaves are simply connected and geodesically complete. If for any spacelike geodesic $\gamma\colon \mathbb{R} \to M$, we have*

$$\left(R^{\mu}{}_{\nu\kappa\rho}R^{\nu}{}_{\mu}{}^{\kappa}{}_{\sigma}v^{\rho}v^{\sigma}\right)\Big|_{\gamma(s)} \xrightarrow{s\to\infty} 0 \tag{2.62}$$

for some unit timelike vector field v, then the manifold has absolute rotation.

Proof. Each spatial leaf is a simply connected geodesically complete flat Riemannian manifold of dimension n, i.e. isometric to n-dimensional Euclidean space.

Due to spatial flatness, the expression on the left-hand side of (2.62) is the same for *any* unit timelike vector field v. In particular, for v rigid, by corollary 2.11 we obtain

$$\left((\nabla_{\kappa}\omega^{\mu}{}_{\nu})\nabla^{\kappa}\omega^{\nu}{}_{\mu}\right)\Big|_{\gamma(s)} \xrightarrow{s\to\infty} 0, \tag{2.63}$$

i.e. $\overset{(n)}{\nabla}\omega$ goes to zero at infinity on each spatial leaf.

Now by the Künzle–Ehlers recovery theorem, on each spatial leaf Σ, ω satisfies $\mathrm{d}(\omega|_{\Sigma}) = 0$ and ${}^{(n)}\delta(\omega|_{\Sigma}) = 0$, i.e. $\omega|_{\Sigma}$ is a harmonic form. This implies that its components ω_{ab} with respect to orthonormal coordinates are harmonic functions, which in turn implies that $\partial_a\omega_{bc}$ are also harmonic functions. Since those vanish at infinity, by the maximum principle for harmonic functions on $\mathbb{R}^n$ they vanish on all of Σ. Thus ω is spatially constant, and we are done. □

2.3. Newton–Cartan gravity as a formal limit of general relativity

In this section, we will show in which sense Newton–Cartan gravity arises as the 'Newtonian limit' of general relativity (GR). We will consider this limit in a purely formal way, namely formulate it as the formal limit $c \to \infty$, where c is the speed of light. To implement this limit, we will expand all objects of GR as formal power series in the parameter c^{-1} – or, more precisely, formal Laurent series, since we will need negative orders of c^{-1} – and consider the behaviour of the terms of order c^0.

Of course, analytically speaking, a 'Taylor expansion' in a dimensionful parameter like c does not make sense (even more so since c is a constant of nature); only for *dimensionless* parameters can a meaningful 'small-parameter approximation' be made. In physical realisations of the limit from GR to Newton–Cartan gravity, this means that the corresponding small parameter has to be chosen as, e.g., the ratio of some typical velocity of the system under consideration to the speed of light. In the following, however, we will forget about these issues and just expand in c^{-1} as a formal parameter.[7] (Ehlers discusses the 'limit' in terms of an actual small parameter approaching zero [Ehl81], and other places in the literature discuss the relationship of formal '$c \to \infty$' limits to actual physical approximations, see, e.g., [TF11].)

The following considerations can be motivated by the observation that the $(n+1)$-dimensional Minkowski metric may be written as

$$\eta = -c^2 \mathrm{d}t^2 + \sum_{a=1}^{n} (\mathrm{d}x^a)^2 \,, \tag{2.64a}$$

and its inverse as

$$\eta^{-1} = \sum_{a=1}^{n} \partial_a \otimes \partial_a - c^{-2} \partial_t \otimes \partial_t \,. \tag{2.64b}$$

Lemma 2.33. *Let (M, g) be a Lorentzian manifold, and assume that the metric g may be expanded as a formal power series[8] in c^{-1} as*

$$g = -c^2 \tau \otimes \tau + \mathrm{O}(c^0) \tag{2.65a}$$

for some nowhere vanishing one-form $\tau \in \Omega^1(M)$. Assume further that the inverse metric has the expansion

$$g^{-1} = h + \mathrm{O}(c^{-2}) \tag{2.65b}$$

for some contravariant degree-2 tensor field h.

[7] Strictly speaking, this means that we are not dealing with tensor fields in the usual sense, but with tensor fields that take values in the field of formal Laurent series $\mathbb{R}((c^{-1}))$ instead of the real numbers. However, for the formal treatment of the theory this does not cause any problems – we just define differentiation of series-valued tensor fields order by order, and demand that equations be satisfied order by order.

[8] As said above, this is a formal *Laurent* series, since we have a term of negative order. We will however continue to use the term 'power series', since most of our series will have terms of just non-negative order.

(i) (M, τ, h) *is a Galilei manifold.*

(ii) Writing $g^{-1} =: h + c^{-2}k + \mathrm{O}(c^{-3})$*, the vector field* $v := -k(\tau, \cdot)$ *is unit timelike in the Newton–Cartan sense (with respect to* τ*).*

(iii) Using v *from above, the metric has the expansion*

$$g = -c^2\tau \otimes \tau + \underset{v}{h} - 2\phi\tau \otimes \tau + \mathrm{O}(c^{-1}) \tag{2.66}$$

for some function ϕ*.*

Proof. We write the expansion of the metric as

$$g = -c^2\tau \otimes \tau + g^{(0)} + \mathrm{O}(c^{-1}) \tag{2.67}$$

for some as of yet unknown tensor field $g^{(0)}$.

The definition of the inverse metric reads $g_{\mu\nu}g^{\nu\rho} = \delta^\rho_\mu$, which with the assumed expansions becomes

$$\delta^\rho_\mu = -c^2\tau_\mu\tau_\nu h^{\nu\rho} + g^{(0)}_{\mu\nu}h^{\nu\rho} - \tau_\mu\tau_\nu k^{\nu\rho} + \mathrm{O}(c^{-1}). \tag{2.68}$$

Comparison of coefficients implies $\tau_\nu h^{\nu\rho} = 0$ and

$$\delta^\rho_\mu = g^{(0)}_{\mu\nu}h^{\nu\rho} - \tau_\mu\tau_\nu k^{\nu\rho}. \tag{2.69}$$

Since the identity map has rank $\dim M =: n + 1$ and the last term in (2.69) has rank 1,[9] the first term on the right-hand side has rank $\geq n$. This implies that h has rank $\geq n$; since it is degenerate in direction τ, it has rank n. To show that (M, τ, h) is a Galilei manifold, it remains to prove that in its n non-degenerate directions h is positive definite. This we will do in a moment.

Contracting (2.69) with τ_ρ, we obtain $-1 = \tau_\nu k^{\nu\rho}\tau_\rho$. This shows on the one hand that $v := -k(\tau, \cdot)$ is unit timelike (in the Newton–Cartan sense), and on the other hand that $g^{-1}(\tau, \tau) = -c^{-2} + \mathrm{O}(c^{-3}) < 0$, i.e. that τ is timelike in the Lorentzian sense with respect to g.

Let now β be any covector such that $h(\beta, \beta) \neq 0$. This implies that β and τ are linearly independent, and therefore the projection $\tilde{\beta} := \beta - \frac{g^{-1}(\tau,\beta)}{g^{-1}(\tau,\tau)}\tau$ of β onto the g^{-1}-orthogonal complement of τ is nonzero. Since τ is timelike, $\tilde{\beta}$ is spacelike (both in the Lorentzian sense). One easily computes that $\tilde{\beta} = \beta + k(\tau, \beta)\tau + \mathrm{O}(c^{-1})$, which implies $0 < g^{-1}(\tilde{\beta}, \tilde{\beta}) = h(\beta, \beta) + \mathrm{O}(c^{-1})$. Thus $h(\beta, \beta) > 0$, and we have shown that h

[9] If it had rank 0, the first term on the right-hand side would be of rank $n + 1$, which can't be the case since h has a degenerate direction and therefore rank at most n.

is positive definite in its non-degenerate directions. Thus now we know that (M, τ, h) is a Galilei manifold.

Now comparing (2.69) to the definition $\delta^\rho_\mu = P^\rho_\mu + \tau_\mu v^\rho$ of the projector with respect to v, we obtain $g^{(0)}_{\mu\nu} h^{\nu\rho} = P^\rho_\mu$. This in turn implies $g^{(0)}_{\mu\nu} P^\nu_\rho = h_{\mu\rho}$, where $h_{\mu\rho}$ are the components of $\underset{v}{h}$. Thus we finally obtain

$$\begin{aligned} g^{(0)}_{\mu\nu} &= g^{(0)}_{\rho\sigma}(P^\rho_\mu + v^\rho \tau_\mu)(P^\sigma_\nu + v^\sigma \tau_\nu) \\ &= h_{\mu\nu} + \underbrace{g^{(0)}(v,v)}_{=:-2\phi} \tau_\mu \tau_\nu \end{aligned} \tag{2.70}$$

and are finished. □

Note that the assumed expansions of the metric and its inverse correspond to the existence of the formal '$c \to \infty$ limits'

$$\lim_{c\to\infty}(c^{-2} g) = -\tau \otimes \tau, \quad \lim_{c\to\infty} g^{-1} = h \tag{2.71}$$

and the statement that the next-to-leading-order terms (c^1 for g, c^{-1} for g^{-1}) vanish.

Construction 2.34. We will now compute the Christoffel symbols of the Lorentzian metric g, i.e. the connection coefficients of the Levi-Civita connection. Using the expansions from lemma 2.33, the Christoffel symbols are

$$\begin{aligned} \overset{g}{\Gamma}{}^\rho_{\mu\nu} &= \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \\ &= \frac{1}{2}\left(h^{\rho\sigma} + c^{-2}k^{\rho\sigma} + \mathrm{O}(c^{-3})\right)\Big(-(c^2 + 2\phi)2(\tau_\sigma \partial_{(\mu}\tau_{\nu)} + \tau_{(\nu}\partial_{\mu)}\tau_\sigma - \tau_{(\mu}\partial_{|\sigma|}\tau_{\nu)}) \\ &\qquad + \partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu} - 4\tau_\sigma \tau_{(\nu}\partial_{\mu)}\phi + 2\tau_\mu \tau_\nu \partial_\sigma \phi + \mathrm{O}(c^{-1})\Big) \\ &= -(c^2 + 2\phi) h^{\rho\sigma}(\tau_{(\nu}\partial_{\mu)}\tau_\sigma - \tau_{(\mu}\partial_{|\sigma|}\tau_{\nu)}) + \frac{1}{2} h^{\rho\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) \\ &\quad + h^{\rho\sigma}\tau_\mu \tau_\nu \partial_\sigma \phi - k^{\rho\sigma}(\tau_{(\nu}\partial_{\mu)}\tau_\sigma - \tau_{(\mu}\partial_{|\sigma|}\tau_{\nu)}) - \underbrace{k^{\rho\sigma}\tau_\sigma}_{=-v^\rho} \partial_{(\mu}\tau_{\nu)} + \mathrm{O}(c^{-1}) \\ &= -[(c^2 + 2\phi)h^{\rho\sigma} + k^{\rho\sigma}]\tau_{(\mu}(\mathrm{d}\tau)_{\nu)\sigma} + v^\rho \partial_{(\mu}\tau_{\nu)} + \frac{1}{2} h^{\rho\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) \\ &\quad + h^{\rho\sigma}\tau_\mu \tau_\nu \partial_\sigma \phi + \mathrm{O}(c^{-1}). \end{aligned} \tag{2.72}$$

Theorem 2.35. *Let (M, g) be a Lorentzian manifold whose metric and inverse metric have expansions as in lemma 2.33.*

(i) (M, τ, h) is a Galilei manifold.

(ii) *The Levi-Civita connection* $\overset{g}{\nabla}$ *of* (M, g) *has a regular formal* $c \to \infty$ *limit, i.e. no terms of negative order in* c^{-1}, *iff* $\mathrm{d}\tau = 0$.

(iii) *If* $\mathrm{d}\tau = 0$, *the formal* $c \to \infty$ *limit of* $\overset{g}{\nabla}$, *i.e. its term of order* c^0, *is a torsion-free Galilei connection* ∇ *on* (M, τ, h). *The curvature tensors of* $\overset{g}{\nabla}$ *and of* ∇ *satisfy the formal limit relation*

$$\lim_{c\to\infty} \overset{g}{R}{}^{\mu}{}_{\nu\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma}\,, \tag{2.73}$$

and ∇ *is Newtonian.*

Proof. (i) was proved in lemma 2.33, and (ii) follows directly from the calculation in construction 2.34 (by decomposing into spacelike and timelike components, one easily checks that $\tau_{(\mu}(\mathrm{d}\tau)_{\nu)\sigma} = 0$ is equivalent to $\mathrm{d}\tau = 0$). Comparing the limiting connection from (2.72) to the general form of Galilei connections from the classification theorem (theorem 1.16), we see that the limit is a torsion-free Galilei connection.

Since the connection coefficients satisfy

$$\overset{g}{\Gamma}{}^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} + \mathrm{O}(c^{-1}) \tag{2.74}$$

and curvature tensor components have the symbolic form

$$R = \partial\Gamma - \partial\Gamma + \Gamma\Gamma - \Gamma\Gamma, \tag{2.75}$$

we directly obtain (2.73). Finally, from

$$\overset{g}{R}{}^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma} = g^{\nu\kappa}\overset{g}{R}{}^{\mu}{}_{\rho\kappa\sigma} = h^{\nu\kappa}R^{\mu}{}_{\rho\kappa\sigma} + \mathrm{O}(c^{-1}) = R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma} + \mathrm{O}(c^{-1}) \tag{2.76}$$

and symmetry in pairs of the Riemannian curvature tensor, we obtain symmetry in pairs for $R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}$, i.e. that ∇ is Newtonian. □

Remark 2.36. Actually, comparing the connection coefficients from (2.72) to the classification theorem, we obtain even more: the Newton–Coriolis form of the limiting Galilei connection ∇ with respect to the unit timelike vector field $v = -k(\tau, \cdot)$ satisfies

$$\tau_{(\mu}\Omega_{\nu)}{}^{\rho} = h^{\rho\sigma}\tau_{\mu}\tau_{\nu}\partial_{\sigma}\phi, \tag{2.77a}$$

from which we can obtain (by decomposing into spacelike and timelike parts)

$$\Omega_{\mu\nu} = 2\tau_{[\mu}\partial_{\nu]}\phi, \tag{2.77b}$$

i.e.

$$\Omega = \tau \wedge \mathrm{d}\phi. \tag{2.77c}$$

Therefore, the unit timelike vector field v obtained from the expansion of the Lorentzian metric is twist-free and has acceleration $\alpha = \mathrm{d}\phi$. Note, however, that v is not necessarily rigid.

Construction 2.37. We now want to look at the Newtonian limit / formal c^{-1} expansion of Einstein's field equation

$$\overset{g}{R}_{\mu\nu} - \frac{1}{2}\overset{g}{R} g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \,. \tag{2.78a}$$

In order to be able to obtain a regular Newtonian limit, we have to consider its trace-reversed form

$$\overset{g}{R}_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{n-1} T g_{\mu\nu}\right), \tag{2.78b}$$

where $n = \dim M - 1 \neq 1$, and $T = g^{\mu\nu} T_{\mu\nu}$ is the trace of the energy–momentum tensor.

We assume the metric and inverse metric to have expansions as in lemma 2.33. Defining the Lorentzian normalised one-form $V_\mu := \tau_\mu / \sqrt{-g^{-1}(\tau,\tau)}$, we assume the energy–momentum tensor to have the expansion

$$T_{\mu\nu} = \hat{\rho} c^2 V_\mu V_\nu + c\Pi_\mu V_\nu + V_\mu c \Pi_\nu + \Sigma_{\mu\nu} \tag{2.79a}$$

where $\hat{\rho}, \Pi, \Sigma$ are, respectively, the mass–energy density, the momentum density / energy current density, and the momentum current density / stress tensor as seen by an observer with spacetime velocity V, which we all assume to be of order c^0. Using $1/\sqrt{-g^{-1}(\tau,\tau)} = c + \mathrm{O}(c^0)$, this implies

$$T_{\mu\nu} = \rho c^4 \tau_\mu \tau_\nu + c^3 (\dots) \tau_\mu \tau_\nu + c^2 [\tau_\mu (\dots) + (\dots) \tau_\nu] + \mathrm{O}(c^1), \tag{2.79b}$$

where ρ is the (rest) mass density, given by the expansion of $\hat{\rho}$ as $\hat{\rho} = \rho + \mathrm{O}(c^{-1})$. Contracting this with $g^{\mu\nu} = h^{\mu\nu} + c^{-2} k^{\mu\nu} + \mathrm{O}(c^{-3})$, the trace of the energy–momentum tensor is

$$T = g^{\mu\nu} T_{\mu\nu} = \rho c^2 k^{\mu\nu} \tau_\mu \tau_\nu + \mathrm{O}(c^1) = -\rho c^2 + \mathrm{O}(c^1), \tag{2.80a}$$

implying

$$T g_{\mu\nu} = \rho c^4 \tau_\mu \tau_\nu + \mathrm{O}(c^3). \tag{2.80b}$$

Inserting (2.79b) and (2.80b) into the trace-reversed Einstein equation (2.78b), we obtain

$$\overset{g}{R}_{\mu\nu} = 8\pi \frac{n-2}{n-1} G \rho \tau_\mu \tau_\nu + \mathrm{O}(c^{-1}). \tag{2.81}$$

For $n = 3$, we thus obtain the Newton–Cartan field equation as the formal $c \to \infty$ limit of the Einstein equation (and the same in higher dimensions, except for a different factor in the field equation).

Theorem 2.38. *Assuming expansions of the Lorentzian metric and inverse metric as in lemma 2.33, regularity of the Lorentzian Levi-Civita connection as $c \to \infty$, and an expansion of the energy–momentum tensor as in (2.79), Newton–Cartan gravity arises as the formal $c \to \infty$ limit of general relativity.*

Proof. In lemma 2.33 and theorem 2.35, we have seen how the formal $c \to \infty$ limit of a Lorentzian spacetime gives rise to a Newtonian spacetime. In construction 2.37, we have shown how the Einstein equation gives rise to the Newton–Cartan field equation. Since the Levi-Civita connection of the Lorentzian spacetime expands to the Newtonian connection plus higher-order terms, Lorentzian geodesics in the limit go over to Newtonian geodesics, i.e. test particle worldlines go over to test particle worldlines. Due to $g_{\mu\nu} = -c^2\tau_\mu\tau_\nu + h_{\mu\nu} - 2\phi\tau_\mu\tau_\nu + \mathrm{O}(c^{-1})$, Lorentzian spacelike vectors are also spacelike in the Newton–Cartan sense, and spatial lengths as defined by g in the limit go over to spatial lengths as defined by h. Finally, ideal clocks in GR measure Lorentzian proper time, which for future-directed worldlines expands as

$$c^{-1}\int_\gamma \sqrt{-g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu} = c^{-1}\int_\gamma \sqrt{c^2\tau_\mu\tau_\nu\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathrm{O}(c^0)} = \int_\gamma \tau + \mathrm{O}(c^{-2}), \tag{2.82}$$

i.e. in the limit becomes (absolute) time in the Newton–Cartan sense. □

Example 2.39. Consider the Schwarzschild metric

$$g = -\left(1 - \frac{2GM}{c^2r}\right)c^2\mathrm{d}t^2 + \left(1 - \frac{2GM}{c^2r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2), \tag{2.83a}$$

with inverse metric given by

$$\begin{aligned} g^{-1} = &-\left(1 - \frac{2GM}{c^2r}\right)^{-1}c^{-2}\partial_t\otimes\partial_t + \left(1 - \frac{2GM}{c^2r}\right)\partial_r\otimes\partial_r \\ &+ r^{-2}(\partial_\theta\otimes\partial_\theta + \sin^{-2}\theta\,\partial_\varphi\otimes\partial_\varphi). \end{aligned} \tag{2.83b}$$

Expanding those in powers of c^{-1} (using the geometric series for the inversion of power series), we obtain

$$g = \underbrace{-c^2\mathrm{d}t^2}_{\overset{!}{=}-c^2\tau\otimes\tau} + \underbrace{\frac{2GM}{r}\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)}_{=g^{(0)}} + \mathrm{O}(c^{-2}), \tag{2.84a}$$

$$\begin{aligned} g^{-1} = &\underbrace{\partial_r\otimes\partial_r + r^{-2}(\partial_\theta\otimes\partial_\theta + \sin^{-2}\theta\,\partial_\varphi\otimes\partial_\varphi)}_{=h} \\ &+ c^{-2}\underbrace{\left(-\partial_t\otimes\partial_t - \frac{2GM}{r}\partial_r\otimes\partial_r\right)}_{=k} + \mathrm{O}(c^{-4}). \end{aligned} \tag{2.84b}$$

We thus see that we may take $\tau = \mathrm{d}t$, and can read off the expansion coefficients $h, g^{(0)}, k$ from lemma 2.33. We see that on the spatial leaves of constant t, the space metric h induces just the usual Euclidean metric in spherical coordinates.

Further, we obtain $v = -k(\tau, \cdot) = \partial_t$, giving $\phi = -\frac{1}{2}g^{(0)}(v, v) = -\frac{GM}{r}$, which leads to $\underset{v}{h} = g^{(0)} + 2\phi\tau \otimes \tau = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2)$ being the Euclidean metric.

Since $v = \partial_t$ and the coordinate components of h do not depend on t, we have $\mathcal{L}_v h = 0$, i.e. v is rigid. According to remark 2.36, it has acceleration $\alpha = \mathrm{d}\phi$ and is twist-free, so we may apply the Trautman recovery theorem and obtain a solution of usual Newtonian gravity with gravitational potential ϕ.

3. Galilei manifolds via principal bundles

Here, we will discuss how we can describe Galilei manifolds and Galilei connections on them in terms of principal bundles.

3.1. The Galilei group

Definition 3.1. The (orthochronous[1]) *homogeneous Galilei group* in $n+1$ dimensions is the semidirect product

$$\mathsf{Gal} = \mathrm{O}(n) \ltimes \mathbb{R}^n, \tag{3.1a}$$

where $\mathrm{O}(n)$ acts on $\mathbb{R}^n$ in the natural way. This means that elements of Gal are pairs (R, k) with $R \in \mathrm{O}(n)$ and $k \in \mathbb{R}^n$, and the group operation is

$$(R, k)(\tilde{R}, \tilde{k}) = (R\tilde{R}, k + R\tilde{k}). \tag{3.1b}$$

It is a Lie group, with Lie algebra the semidirect sum[2]

$$\mathfrak{gal} = \mathfrak{so}(n) \oplus \mathbb{R}^n. \tag{3.2a}$$

This means that as a vector space, $\mathfrak{gal}$ is the direct sum of $\mathfrak{so}(n)$ and $\mathbb{R}^n$, while the Lie bracket is given by

$$[(X, k), (\tilde{X}, \tilde{k})] = ([X, \tilde{X}], X\tilde{k} - \tilde{X}k) \tag{3.2b}$$

[1]Instead, we could also consider the full homogeneous Galilei group, including time reversal, or restrict to the proper (without reflections, but including time reversal) or proper orthochronous Galilei group (which is the connected component of the identity). However, our definition of Galilei manifolds from chapter 1 comes with a choice of time orientation and without the need for space orientability; thus the orthochronous group is the one we will need in the following. Of course, it is a matter of convention what precise definition of a Galilei manifold one chooses.

[2]Note that the Lie algebras of $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ coincide, the latter group being the connected component of the identity of the former.

The homogeneous Galilei group acts on $\mathbb{R}^{n+1}$ via

$$(R,k)(s,y) = (s, Ry + sk), \tag{3.3}$$

where $s \in \mathbb{R}$ and $y \in \mathbb{R}^n$. In this way, it can be viewed as a subgroup of $\mathrm{GL}(n+1)$.

Interpreting s as the time and y as the position of an event in 'standard Newtonian spacetime', we see that the $\mathrm{O}(n)$ part of Gal corresponds to improper rotations of space, while the $\mathbb{R}^n$ part corresponds to Galilei boosts.

Notation 3.2. When understanding $\mathbb{R}^{n+1}$ as the space on which the Galilei group acts according to (3.3), we will often write elements of $\mathbb{R}^{n+1}$ as $y = (y^t, y^a)$, i.e. use t as a 'temporal' index and lowercase Latin letters from the beginning of the alphabet as 'spatial' indices running from 1 to n, as we did for labelling the vectors of a Galilei basis (definition 1.20), or in index notation in adapted coordinates for Galilei manifolds. Capital Latin indices run through the full range $\{t, 1, \ldots, n\}$, i.e. $y = (y^t, y^a) = (y^A)$.

Remark 3.3. Writing the action (3.3) of Gal on $\mathbb{R}^{n+1}$ in matrix form, we see that when viewing Gal as a subgroup of $\mathrm{GL}(n+1)$, the matrix corresponding to a Galilei group element $(R,k) \in \mathrm{Gal}$ is

$$\begin{pmatrix} 1 & 0 \\ k & R \end{pmatrix} \in \mathrm{GL}(n+1). \tag{3.4a}$$

Therefore, the condition for a matrix $A \in \mathrm{GL}(n+1)$ to be an element of Gal can be written in components as

$$A^t{}_t = 1, \quad A^t{}_a = 0, \quad (A^a{}_b) \in \mathrm{O}(n). \tag{3.4b}$$

Viewing the homogeneous Galilei Lie algebra $\mathfrak{gal}$ as a subalgebra of the matrix Lie algebra $\mathfrak{gl}(n+1)$ of all $(n+1)\times(n+1)$ matrices, the abstract Lie algebra element $(X,k) \in \mathfrak{gal} = \mathfrak{so}(n) \oplus \mathbb{R}^n$ is represented by the matrix

$$\begin{pmatrix} 0 & 0 \\ k & X \end{pmatrix} \in \mathfrak{gl}(n+1). \tag{3.5a}$$

In components, the condition for a matrix $Y \in \mathfrak{gl}(n+1)$ to lie in $\mathfrak{gal}$ therefore reads

$$Y^t{}_A = 0, \quad (Y^a{}_b) \in \mathfrak{so}(n). \tag{3.5b}$$

Definition 3.4. The (orthochronous) *inhomogeneous Galilei group* in $n+1$ dimensions is the semidirect product

$$\mathrm{IGal} = \mathrm{Gal} \ltimes \mathbb{R}^{n+1}, \tag{3.6}$$

where Gal acts on $\mathbb{R}^{n+1}$ as in (3.3). Here, the $\mathbb{R}^{n+1}$ part corresponds to spacetime translations.[3]

[3]Note that the decomposition of the inhomogeneous group into the homogeneous subgroup and the

3.2. Galilei structures

Lemma 3.5. *Let (M, τ, h) be a Galilei manifold and (e_A) a Galilei basis at $p \in M$. Another basis $(\tilde{\mathrm{e}}_A)$ of T_pM is a Galilei basis iff the basis change matrix $A \in \mathsf{GL}(n+1)$, defined by $\tilde{\mathrm{e}}_A = A^B{}_A \mathrm{e}_B$, is an element of the homogeneous Galilei group* Gal *(understood as a subgroup of* $\mathsf{GL}(n+1)$ *via (3.3)).*

Proof. We first recall that if two bases of a vector space are related by $\tilde{\mathrm{e}}_A = A^B{}_A \mathrm{e}_B$, then the dual bases are related by $\tilde{\mathrm{e}}^A = (A^{-1})^A{}_B \mathrm{e}^B$. Of course, we can also invert the original relation to $\mathrm{e}_A = (A^{-1})^B{}_A \tilde{\mathrm{e}}_B$. The second basis $(\tilde{\mathrm{e}}_A)$ of T_pM being a Galilei basis means that

$$\tilde{\mathrm{e}}^t = \tau|_p = \mathrm{e}^t, \quad \delta^{ab} \tilde{\mathrm{e}}_a \otimes \tilde{\mathrm{e}}_b = h|p = \delta^{ab} \mathrm{e}_a \otimes \mathrm{e}_b \,. \tag{3.7}$$

Expressing $\tilde{\mathrm{e}}^t$ in terms of the e^A in the first equation and the e_a in terms of the $\tilde{\mathrm{e}}_A$ in the second one, these are equivalent to

$$(A^{-1})^t{}_t \mathrm{e}^t + (A^{-1})^t{}_a \mathrm{e}^a = \mathrm{e}^t, \tag{3.8a}$$

i.e.

$$(A^{-1})^t{}_t = 1, \quad (A^{-1})^t{}_a = 0, \tag{3.8b}$$

and

$$\begin{aligned} \delta^{ab} \tilde{\mathrm{e}}_a \otimes \tilde{\mathrm{e}}_b &= \delta^{ab} \left((A^{-1})^t{}_a \tilde{\mathrm{e}}_t + (A^{-1})^c{}_a \tilde{\mathrm{e}}_c \right) \otimes \left((A^{-1})^t{}_b \tilde{\mathrm{e}}_t + (A^{-1})^d{}_b \tilde{\mathrm{e}}_d \right) \\ &= (A^{-1})^c{}_a \delta^{ab} (A^{-1})^d{}_b \tilde{\mathrm{e}}_c \otimes \tilde{\mathrm{e}}_d \\ &= (A^{-1})^a{}_c \delta^{cd} (A^{-1})^b{}_d \tilde{\mathrm{e}}_a \otimes \tilde{\mathrm{e}}_b \,, \end{aligned} \tag{3.8c}$$

i.e.

$$\delta^{ab} = (A^{-1})^a{}_c \delta^{cd} (A^{-1})^b{}_d \iff ((A^{-1})^a{}_b) \in \mathsf{O}(n). \tag{3.8d}$$

Comparing to remark 3.3, we see that this means that A^{-1} is an element of Gal, which is of course equivalent to A being an element of Gal. □

translation normal subgroup depends on the choice of an arbitrary origin in spacetime. It is only with respect to this origin that we may label a general inhomogeneous Galilei transformation as homogeneous (namely if it fixes the origin). However, since in the following we need to interpret 'standard Newtonian spacetime' as the vector space $\mathbb{R}^{n+1}$, and not as an affine space over an $(n+1)$-dimensional vector space with some extra structure, we introduced 'the' Galilei group in the above way, which from a more fundamental perspective may be seen as kind of 'too concrete'.

Definition 3.6. Let (M, τ, h) be a Galilei manifold. The set of all Galilei bases at a point $p \in M$ will be denoted by

$$G_p(M) := \{(\mathrm{e}_A) : (\mathrm{e}_A) \text{ is a Galilei basis of } T_pM\}. \tag{3.9}$$

The disjoint union

$$G(M) := \coprod_{p \in M} G_p(M) \tag{3.10}$$

is the *Galilei frame bundle* of (M, τ, h).[4] We have a natural projection $\pi \colon G(M) \to M$ sending any Galilei basis to the point at which it lives. According to lemma 3.5, by basis change we can define a right action of the homogeneous Galilei group Gal on $G(M)$ which is free and whose orbits are the fibres $G_p(M)$. Spelled out, it reads

$$(\mathrm{e}_t, \mathrm{e}_a) \cdot (R, k) = (\mathrm{e}_t + k^b \mathrm{e}_b, \mathrm{e}_b R^b{}_a) \tag{3.11}$$

for $(\mathrm{e}_A) \in G(M)$ and $(R, k) \in \mathrm{Gal}$.

$G(M)$ can be made into a smooth principal bundle over M in essentially the same way as the linear frame bundle $F(M)$: around any point $p \in M$, we can find an open neighbourhood $U \subset M$ and smooth vector fields e_A defined on U such that evaluated at any point in U, they form a Galilei basis.[5] Demanding that the corresponding bijective maps

$$G(M)|_U := \pi^{-1}(U) \to U \times \mathrm{Gal}, \tag{3.12a}$$
$$G_q(M) \ni (\mathrm{e}_B|_q A^B{}_A) \mapsto (q, A) \tag{3.12b}$$

for all such locally defined Galilei basis fields be homeomorphisms, we obtain a topology on $G(M)$, and then demanding that they all be diffeomorphisms, we obtain a smooth structure on $G(M)$. These maps then are local trivialisations of the Galilei frame bundle, and it is a principal Gal-bundle over M.

A local section $(\mathrm{e}_A) \in \Gamma(U, G(M))$ of the Galilei frame bundle, i.e. $n+1$ locally defined vector fields that at any point are a Galilei basis, is a *(local) Galilei frame* on (M, τ, h).

Notation 3.7. We will usually denote the unit timelike field of a Galilei frame as $\mathrm{e}_t =: v$, as was our usual notation for (local) choices of unit timelike future-directed vector fields.

[4]Of course, the Galilei frame bundle depends on τ and h, but we will not acknowledge this in our notation.

[5]Such a basis of vector fields may be constructed as follows: we choose *any* frame of vector fields $(\tilde{\mathrm{e}}_A)$ on some neighbourhood of p such that $\tilde{\mathrm{e}}_t =: \mathrm{e}_t$ is a unit timelike future-directed field, then project the other fields $\tilde{\mathrm{e}}_a$ onto space along $\tilde{\mathrm{e}}_t$ and apply the Gram–Schmidt orthonormalisation process with respect to ${}^{(n)}h$ to them. Due to smoothness of ${}^{(n)}h$, the resulting fields e_a are smooth; and by construction, (e_A) will then at each point be a Galilei basis.

Definition 3.8. Let G be a Lie group and $\hat{\pi}\colon P \to M$ a principal G-bundle. A *reduction of the structure group* of P to a Lie subgroup (i.e. a closed subgroup) $H \subset G$ is an embedded submanifold $Q \subset P$ that is invariant (as a set) under the action of H, such that with the restricted projection

$$\pi := \hat{\pi}|_Q : Q \to M \tag{3.13}$$

and the restricted action of H it is a principal H-bundle.[6]

Intuitively, one can imagine the reduced bundle Q 'sitting inside' the larger bundle P, such that locally with respect to local trivialisations, this looks just like H sitting inside of G.

Construction 3.9. For a Galilei manifold (M, τ, h), by construction the Galilei frame bundle $G(M)$ is a reduction of the structure group of the linear frame bundle $F(M)$ from $\mathsf{GL}(n+1)$ to the homogeneous Galilei group Gal, viewed as a subgroup of $\mathsf{GL}(n+1)$.

Conversely, for an $(n+1)$-dimensional manifold M, given a reduction $G(M)$ of the structure group of the frame bundle $F(M)$ to Gal, there are a unique clock form τ and space metric h making M into a Galilei manifold such that $G(M)$ is the Galilei frame bundle of (M, τ, h). Namely, given any local section $(\mathrm{e}_A) \in \Gamma(U, G(M)) \subset \Gamma(U, F(M))$, we can define h and τ on U by $h := \delta^{ab}\mathrm{e}_a \otimes \mathrm{e}_b$ and $\tau := \mathrm{e}^t$, where (e^A) is the dual basis to (e_A). Smoothness of τ and h thus defined is immediate, and that they indeed define a Galilei manifold follows directly by linear independence of the e_a and by definition of the dual basis. Finally, τ and h are well-defined, i.e. independent of the choice of local section, due to Gal-invariance of $G(M)$: any other local section $(\tilde{\mathrm{e}}_A)$ is related to (e_A) by a Galilei transformation at each point, so by lemma 3.5 $(\tilde{\mathrm{e}}_A|_p)$ is also a Galilei basis with respect to τ and h, i.e. $\tau = \tilde{\mathrm{e}}^t$ and $h = \delta^{ab}\tilde{\mathrm{e}}_a \otimes \tilde{\mathrm{e}}_b$.

Put differently, the data τ, h that make a manifold into a Galilei manifold are equivalent to a choice of which linear frames are Galilei frames.

[6] In general, a reduction of P along a Lie group homomorphism $\phi\colon H \to G$ is a principal H-bundle $\pi\colon Q \to M$ together with a smooth map $f\colon Q \to P$ such that

$$\hat{\pi} \circ f = \pi \tag{3.14a}$$

and

$$f(q \cdot h) = f(q) \cdot \phi(h) \tag{3.14b}$$

for all $q \in Q$, $h \in H$. However, if λ is the embedding of a Lie subgroup, then since f is compatible with local trivialisations (by compatibility with the actions), this implies that f is an embedding and we arrive, up to isomorphism, at the picture from our definition.

Definition 3.10. A *Galilei structure* on an $(n+1)$-dimensional smooth manifold M is a reduction $G(M)$ of the structure group of the linear frame bundle $F(M)$ from $\mathrm{GL}(n+1)$ to the homogeneous Galilei group Gal.

Notation 3.11. Since the Galilei frame bundle $G(M)$ is a reduction of the structure group of the linear frame bundle $F(M)$, we obtain the tangent bundle TM as an associated vector bundle,

$$TM \cong G(M) \times_{\mathrm{Gal}} \mathbb{R}^{n+1}. \tag{3.15}$$

The isomorphism is given by the canonical solder form $\theta \in \Omega^1(M, G(M) \times_{\mathrm{Gal}} \mathbb{R}^{n+1}) = \mathrm{Hom}(TM, G(M) \times_{\mathrm{Gal}} \mathbb{R}^{n+1})$,

$$\theta_p(v^A \mathrm{e}_A|_p) = [(\mathrm{e}_A|_p), (v^A)]. \tag{3.16}$$

Using the representations of Gal induced by tensor representations of $\mathrm{GL}(n+1)$, we also obtain all tensor bundles of M as vector bundles associated to the Galilei frame bundle; i.e. any tensor (field) on M can be represented, with respect to a chosen local Galilei frame, by its components with respect to this frame.

When a Galilei frame (e_A) is chosen, we denote tensor components with respect to this frame by the same kind of indices $(A) = (t, a)$ that we use for the frame vectors. For example, we may locally write a vector field $X \in \Gamma(TM)$ as

$$X = X^A \mathrm{e}_A \text{ with } X^A = \mathrm{e}^A(X) = \mathrm{e}^A_\mu X^\mu, \tag{3.17}$$

or a one-form $\alpha \in \Omega^1(M)$ as

$$\alpha = \alpha_A \mathrm{e}^A \text{ with } \alpha_A = \alpha(\mathrm{e}_A) = \mathrm{e}^\mu_A \alpha_\mu. \tag{3.18}$$

So from a purely notational point of view, we can use the components e^μ_A of a Galilei frame and e^A_μ of the dual frame to 'convert' tensor indices from 'coordinate indices' μ to 'Galilei frame indices' A and back.

Construction 3.12. We now want to consider principal connections $\boldsymbol{\omega}$ on the Galilei frame bundle $G(M)$ of a Galilei manifold (M, τ, h). The connection form $\boldsymbol{\omega}$ is a one-form on $G(M)$ with values in the Galilei Lie algebra $\mathfrak{gal} = \mathfrak{so}(n) \oplus \mathbb{R}^n$, i.e. it can be decomposed as

$$\boldsymbol{\omega} = (\boldsymbol{\omega}^a{}_b, \boldsymbol{\varpi}^a) \in \Omega^1(G(M), \mathfrak{so}(n) \oplus \mathbb{R}^n), \tag{3.19}$$

with an $\mathfrak{so}(n)$-valued part $(\boldsymbol{\omega}^a{}_b)$ and an $\mathbb{R}^n$-valued part $(\boldsymbol{\varpi}^a)$. The local connection form[7] ω with respect to a Galilei frame (e_A) on $U \subset M$, i.e. the pullback of the

[7]Note that we denote the connection form on the principal bundle by boldface letters, and local connection forms by the corresponding non-boldface letters. Note also that we denote principal connections by the same letter ω that we earlier used to refer to the twist of a vector field. The meaning should always be clear from context.

connection form $\boldsymbol{\omega} \in \Omega^1(G(M), \mathfrak{gal})$ along (e_A), is then a $\mathfrak{gal}$-valued one-form on U, which we can similarly decompose as

$$\omega = ({\omega^a}_b, \varpi^a) \in \Omega^1(U, \mathfrak{so}(n) \oplus \mathbb{R}^n). \tag{3.20}$$

Via remark 3.3, we may also understand this as a matrix-valued one-form $({\omega^A}_B) \in \Omega^1(U, \mathfrak{gl}(n+1))$ with

$${\omega^t}_A = 0, \quad ({\omega^a}_b) \in \Omega^1(U, \mathfrak{so}(n)), \quad {\omega^a}_t = \varpi^a. \tag{3.21}$$

The covariant derivative operator ∇ that is induced by $\boldsymbol{\omega}$ on the tangent bundle $TM \cong G(M) \times_{\mathrm{Gal}} \mathbb{R}^{n+1}$ then acts as follows:[8] for a vector field $X = X^A \mathrm{e}_A$, we have

$$(\nabla X)^A = \mathrm{d}X^A + {\omega^A}_B \otimes X^B, \tag{3.23a}$$

i.e.

$$(\nabla X)^t = \mathrm{d}X^t, \tag{3.23b}$$
$$(\nabla X)^a = \mathrm{d}X^a + {\omega^a}_b \otimes X^b + \varpi^a \otimes X^t. \tag{3.23c}$$

The covariant derivative induced on T^*M acts on one-forms $\alpha = \alpha_A \mathrm{e}^A$ as

$$(\nabla \alpha)_A = \mathrm{d}\alpha_A - {\omega^B}_A \otimes \alpha_B\ , \tag{3.24a}$$

i.e.

$$(\nabla \alpha)_t = \mathrm{d}\alpha_t - \varpi^a \otimes \alpha_a\ , \tag{3.24b}$$
$$(\nabla \alpha)_a = \mathrm{d}\alpha_a - {\omega^b}_a \otimes \alpha_b\ . \tag{3.24c}$$

On higher-degree tensors, the induced covariant derivative acts analogously: we get a local connection form acting on each contravariant index and minus a local connection form acting on each covariant index (since that's the tensor representation of the Lie algebra $\mathfrak{gl}(n+1)$).

Now since for a Galilei frame we have $\tau = \mathrm{e}^t$, i.e. $\tau_A = \delta^t_A$, we can compute

$$\begin{aligned}(\nabla \tau)_A &= \mathrm{d}\tau_A - {\omega^B}_A \otimes \tau_B \\ &= 0 - {\omega^B}_A \delta^t_B = {\omega^t}_A \overset{(3.21)}{=} 0;\end{aligned} \tag{3.25a}$$

[8]This follows from the general formula how a principal connection induces a covariant derivative on associated vector bundles: given some representation ρ of Gal on a vector space V, on a local section $[(\mathrm{e}_A), v]$ of the associated vector bundle $G(M) \times_\rho V$ the induced covariant derivative acts as

$$\overset{\omega}{\nabla}[(\mathrm{e}_A), v] = [(\mathrm{e}_A), \mathrm{d}v + \dot{\rho}_\omega(v)] \tag{3.22}$$

where $\dot{\rho}\colon \mathfrak{gal} \to \mathfrak{gl}(V)$ is the Lie algebra representation induced by ρ.

and similarly from $h = \delta^{ab} \mathrm{e}_a \otimes \mathrm{e}_b$, i.e. $h^{AB} = \delta^A_a \delta^B_b \delta^{ab}$, we get

$$(\nabla h)^{AB} = \underbrace{\mathrm{d}h^{AB}}_{=0} + \omega^A{}_C \otimes h^{CB} + \omega^B{}_C \otimes h^{AC}, \tag{3.25b}$$

i.e.

$$(\nabla h)^{tA} = (\nabla h)^{At} = \omega^t{}_C \otimes h^{AC} \overset{(3.21)}{=} 0, \tag{3.25c}$$

$$(\nabla h)^{ab} = \delta^{cb} \omega^a{}_c + \delta^{ac} \omega^b{}_c \overset{(3.21)}{=} 0. \tag{3.25d}$$

This means that the induced ∇ is a Galilei connection on (M, τ, h).

Conversely, given a Galilei connection ∇ on (M, τ, h), by going backwards through the above calculations, we see that its local connection form $(\omega^A{}_B) \in \Omega^1(U, \mathfrak{gl}(n+1))$ with respect to a Galilei frame (e_A) on U, defined by

$$\nabla \mathrm{e}_A =: \omega^B{}_A \otimes \mathrm{e}_B \,, \tag{3.26}$$

satisfies (3.21), i.e. takes values in $\mathfrak{gal}$. Thus, ∇ gives rise to a principal connection $\boldsymbol{\omega}$ on the Galilei frame bundle $G(M)$ by which it is induced.

So Galilei connections on a Galilei manifold are essentially the same as principal connections on the Galilei frame bundle:

Proposition 3.13. *Let (M, τ, h) be a Galilei manifold with Galilei frame bundle $G(M)$.*

(i) Given a principal connection $\boldsymbol{\omega}$ on $G(M)$, the induced covariant derivative operator ∇ on the tangent bundle $TM \cong G(M) \times_{\mathrm{Gal}} \mathbb{R}^{n+1}$ is a Galilei connection on (M, τ, h).

(ii) Conversely, given a Galilei connection ∇ on (M, τ, h), it is induced by a (unique) principal connection $\boldsymbol{\omega}$ on $G(M)$. □

Proposition 3.14. *Let (M, τ, h) be a Galilei manifold with Galilei frame bundle $G(M)$, and ∇ a Galilei connection on (M, τ, h) with corresponding principal connection $\boldsymbol{\omega}$ on $G(M)$. Let $(\mathrm{e}_A) = (v, \mathrm{e}_a)$ be a local Galilei frame.*

(i) The Newton–Coriolis form of ∇ with respect to v can be written in terms of the 'boost part' of the local connection form and the dual frame as

$$\Omega = \delta_{ab} \varpi^a \wedge \mathrm{e}^b, \tag{3.27a}$$

i.e. in components

$$\Omega_{\mu\nu} = \delta_{ab} 2 \varpi_{[\mu}{}^a \mathrm{e}^b_{\nu]} \,. \tag{3.27b}$$

Writing $\varpi_\mu{}^\nu := \varpi_\mu{}^a \mathrm{e}_a^\nu$, *this may be expressed as*

$$\Omega_{\mu\nu} = 2\varpi_{[\mu\nu]}\ , \tag{3.28}$$

i.e. in some sense the Newton–Coriolis form is the antisymmetrisation of the boost part of the local connection form.

(ii) *Consider the associated bundle* $E := G(M) \times_{\mathrm{Gal}} \mathbb{R}^{n+1}$, *which is isomorphic to* TM *via its canonical solder form* $\theta \in \Omega^1(M, E)$. *Taking the exterior covariant derivative of the solder form with respect to* $\boldsymbol{\omega}$ *yields the torsion* $T \in \Omega^2(M, TM) \cong \Omega^2(M, E)$ *of* ∇, *i.e. we have*

$$T = \mathrm{d}^\omega \theta. \tag{3.29}$$

(This is Cartan's first structure equation, *and is true for any connection on the linear frame bundle or a reduction thereof.)*

Proof. (i) Rewriting the definition of the Newton–Coriolis form with proposition 1.22 (ii) and the definition of the local connection form, we obtain

$$\begin{aligned} \Omega_{\mu\nu} &= 2(\nabla_{[\mu} v^\rho) h_{\nu]\rho} \\ &= 2\,\underbrace{\left(\nabla_{[\mu} \mathrm{e}_t^\rho\right)}_{=\omega_{[\mu|}{}^A{}_t \mathrm{e}^\rho_{A|} = \varpi_{[\mu}{}^c \mathrm{e}^\rho_{|c|}} \delta_{ab} \mathrm{e}^a_{\nu]} \mathrm{e}^b_\rho \\ &= 2\varpi_{[\mu|}{}^c \mathrm{e}^\rho_c \delta_{ab} \mathrm{e}^a_{|\nu]} \mathrm{e}^b_\rho\,. \end{aligned} \tag{3.30}$$

Combining $\mathrm{e}^\rho_c \mathrm{e}^b_\rho = \delta^b_c$, this gives

$$\Omega_{\mu\nu} = 2\varpi_{[\mu}{}^c \mathrm{e}^a_{\nu]} \delta_{ab} \delta^b_c = 2\varpi_{[\mu}{}^b \mathrm{e}^a_{\nu]} \delta_{ab}\ ; \tag{3.31}$$

instead using $\varpi_\mu{}^c \mathrm{e}^\rho_c = \varpi_\mu{}^\rho$, we get

$$\Omega_{\mu\nu} = 2\varpi_{[\mu}{}^\rho \underbrace{\mathrm{e}^a_{\nu]} \delta_{ab} \mathrm{e}^b_\rho}_{=h_{\nu]\rho}} = 2\varpi_{[\mu\nu]}\ . \tag{3.32}$$

(ii) Writing the solder form as $\theta = [(\mathrm{e}_A), (\theta^A)]$ in terms of its local representative $(\theta^A) \in \Omega^1(U, \mathbb{R}^{n+1})$, the local representative of its exterior covariant derivative $\mathrm{d}^\omega\theta = [(\mathrm{e}_A), ((\mathrm{d}^\omega\theta)^A)]$ is given by

$$(\mathrm{d}^\omega\theta)^A = \mathrm{d}\theta^A + \omega^A{}_B \wedge \theta^B, \tag{3.33}$$

in terms of the local connection form. By definition of the canonical solder form, its local representative is just the dual frame, $\theta^A = \mathrm{e}^A$. Therefore, applied to two vector fields X, Y, we obtain

$$\begin{aligned}
(\mathrm{d}^\omega\theta)^A(X,Y) &= \mathrm{d}\theta^A(X,Y) + (\omega^A{}_B \wedge \theta^B)(X,Y) \\
&= X(\theta^A(Y)) - Y(\theta^A(X)) - \theta^A([X,Y]) \\
&\quad + \omega^A{}_B(X)\theta^B(Y) - \omega^A{}_B(Y)\theta^B(X) \\
&= \mathrm{d}Y^A(X) - \mathrm{d}X^A(Y) - [X,Y]^A + \omega^A{}_B(X)Y^B - \omega^A{}_B(Y)X^B \\
&= (\nabla Y)^A(X) - (\nabla X)^A(Y) - [X,Y]^A \\
&= (\nabla_X Y - \nabla_Y X - [X,Y])^A \\
&= (T(X,Y))^A,
\end{aligned} \tag{3.34}$$

where we used the coordinate-free formula for the exterior derivative and the fact that $\theta^A(X) = \mathrm{e}^A(X) = X^A$. □

Using the identification of the torsion as the exterior covariant derivative of the canonical solder form, we obtain yet another way to show that the temporal torsion of a Galilei connection on (M, τ, h) is $\mathrm{d}\tau$ (proposition 1.6 (i)):

$$T^t = (\mathrm{d}^\omega\theta)^t = \underbrace{\mathrm{d}\theta^t}_{=\mathrm{d}e^t=\mathrm{d}\tau} + \underbrace{\omega^t{}_B}_{\overset{(3.21)}{=}0} \wedge \theta^B = \mathrm{d}\tau \tag{3.35}$$

The general properties of the curvature tensor of a Galilei manifold (proposition 1.6 (ii)), namely the vanishing of the temporal curvature and the antisymmetry of the curvature tensor with second index raised with h, also have a simple interpretation in terms of the principal bundle formalism: in terms of the local curvature form $R \in \Omega^2(U, \mathfrak{gl}(n+1))$ of ω, the curvature tensor is

$$\frac{1}{2}R^\mu{}_{\nu\rho\sigma}\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma = \mathrm{e}^\mu_A \mathrm{e}^B_\nu R^A{}_B\,, \tag{3.36}$$

and so these properties of $R^\mu{}_{\nu\rho\sigma}$ are equivalent (via remark 3.3) to the local curvature form R taking values in $\mathfrak{gal} \subset \mathfrak{gl}(n+1)$.

Definition 3.15. Any change of local Galilei frame has the form

$$(v, \mathrm{e}_a) \to (\tilde{v}, \tilde{\mathrm{e}}_a) = (v, \mathrm{e}_a) \cdot (R, k)^{-1} \tag{3.37a}$$

for a local Gal-valued function (R, k). Such a change of frame is called a *local Galilei transformation*. Spelling out the action via (3.11) and using $(R,k)^{-1} = (R^{-1}, -R^{-1}k)$, it reads

$$(v, \mathrm{e}_a) \to (\tilde{v}, \tilde{\mathrm{e}}_a) = \left(v - \mathrm{e}_b (R^{-1})^b{}_a k^a, \mathrm{e}_b (R^{-1})^b{}_a\right). \tag{3.37b}$$

This defines a left action on local Galilei frames by the group of local Gal-valued functions.

Note that local Galilei *boosts*, i.e. local Galilei transformations with $(R, k) = (\mathbb{1}, k)$ for some $\mathbb{R}^n$-valued function k, have the form

$$(v, \mathrm{e}_a) \to (\tilde{v}, \tilde{\mathrm{e}}_a) = (v - \mathrm{e}_a k^a, \mathrm{e}_a). \tag{3.38}$$

Comparing this to (1.34), we see that this is a Milne boost of the reference vector field v with spacelike Milne boost vector field $k^\mu = k^a \mathrm{e}_a^\mu$ – *Milne boosts are local Galilei boosts*!

Under local Galilei transformations of the frame, the dual frame and the local connection forms transform as follows:

Proposition 3.16. *(i) Under a local Galilei transformation (3.37) which is purely rotational, i.e. with $k = 0$, the dual frame and local connection forms transform as*

$$(\tau, \mathrm{e}^a) \to (\tau, R^a{}_b \mathrm{e}^b), \tag{3.39a}$$

$$\omega^a{}_b \to R^a{}_c \omega^c{}_d (R^{-1})^d{}_b + R^a{}_c \mathrm{d}(R^{-1})^c{}_b\,, \tag{3.39b}$$

$$\varpi^a \to R^a{}_b \varpi^b. \tag{3.39c}$$

(ii) Under a Milne boost, i.e. a local Galilei boost with parameter k, the dual frame and local connection forms transform as

$$(\tau, \mathrm{e}^a) \to (\tau, \mathrm{e}^a + k^a \tau), \tag{3.40a}$$

$$\omega^a{}_b \to \omega^a{}_b\,, \tag{3.40b}$$

$$\varpi^a \to \varpi^a - \mathrm{d}k^a - \omega^a{}_b k^b. \tag{3.40c}$$

Proof. This follows easily from the general transformation of the dual basis under a change of basis, and from the general formula

$$(\sigma \cdot g^{-1})^* \boldsymbol{\omega} = g \cdot (\sigma^* \boldsymbol{\omega}) \cdot g^{-1} + g \cdot \mathrm{d}(g^{-1}) \tag{3.41}$$

for the change of local connection form under a change of local section for principal bundles whose structure group is a matrix Lie group. Details are left as an exercise. □

Using these results, we can now actually *derive* the transformation behaviour of the covariant space metric $\underset{v}{h}$ and the Newton–Coriolis form Ω under Milne boosts, which was just stated out of nowhere in the statement of proposition 1.19: we express $\underset{v}{h}$ and Ω in terms of e^A and ϖ (according to (1.38b) and (3.27a)), and use the transformation behaviour (3.40) of those. Here, we also leave the details as an exercise.

4. Semidirect extensions of principal bundles

In this chapter, we will develop a natural construction that, given a semidirect product $H \ltimes N$ of two Lie groups, allows the extension of a principal bundle with structure group the non-normal subgroup H of the product to a larger bundle with structure group the product $H \ltimes N$, and classify connections on the extended bundle. This theory will be used in the next chapter for the description of Newton–Cartan gravity in terms of the so-called Bargmann group.

Construction 4.1. Let H, N be Lie groups and $\rho \colon H \to \mathrm{Aut}(N)$ a Lie group homomorphism[1]. Let $P \overset{\pi}{\to} M$ be a principal H-bundle. We can extend P to a principal $H \ltimes N$-bundle[2] $Q \overset{\hat{\pi}}{\to} M$ as follows: denoting by $\tilde{\rho} \colon H \to \mathrm{Diff}(H \ltimes N)$ the natural left action of H on $H \ltimes N$ by multiplication, i.e.

$$\tilde{\rho}_{h_2}(h_1, n) := (h_2, \mathrm{e}_N)(h_1, n) = (h_2 h_1, \rho_{h_2}(n)), \tag{4.1a}$$

we define Q as the associated bundle

$$Q := P \times_{\tilde{\rho}} (H \ltimes N). \tag{4.1b}$$

The natural right action of $H \ltimes N$ on itself (by multiplication) induces a free right action on Q which is transitive on the fibres and compatible with the local trivialisations, thus making Q into a principal bundle as desired. Explicitly, the action is given by

$$[p, (h, n)] \cdot (\tilde{h}, \tilde{n}) := [p, (h, n)(\tilde{h}, \tilde{n})] = [p, (h\tilde{h}, n\rho_h(\tilde{n}))]. \tag{4.2}$$

We also obtain natural bundle homomorphisms $Q \underset{\gamma}{\overset{\beta}{\rightleftarrows}} P$ satisfying $\beta \circ \gamma = \mathrm{id}_P$, namely

$$\gamma(p) = [p, (\mathrm{e}_H, \mathrm{e}_N)], \quad \beta([p, (h, n)]) = ph\,. \tag{4.3}$$

[1]We only need ρ to be a group homomorphism that is smooth in the sense that the joint map $H \times N \ni (h, n) \mapsto \rho_h(n) \in N$ is smooth, which makes sense also if $\mathrm{Aut}(N)$ is not a Lie group (which is the case if the group of connected components of N is not finitely generated).

[2]Here we mean of course the semidirect product with respect to ρ, omitting it from the notation.

By construction, with respect to local trivialisations $P \xrightarrow{\gamma} Q$ looks like the inclusion $H \hookrightarrow H \ltimes N$, such that it really exhibits Q as a principal bundle extension of P (i.e. P is a reduction of the structure group of Q from $H \ltimes N$ to H.)

We now want to classify connections on the extended bundle Q.

Lemma 4.2. *We use the notation from construction 4.1. Let $\hat{\boldsymbol{\omega}} \in \Omega^1(Q, \mathfrak{h} \oplus \mathfrak{n})$ be a connection on Q. We decompose its pullback along γ as $\gamma^*\hat{\boldsymbol{\omega}} = (\boldsymbol{\omega}, \boldsymbol{\theta})$ with $\boldsymbol{\omega} \in \Omega^1(P, \mathfrak{h})$ and $\boldsymbol{\theta} \in \Omega^1(P, \mathfrak{n})$. Then $\boldsymbol{\omega}$ is a connection and $\boldsymbol{\theta}$ is a $\dot{\rho}$-tensorial form on P, where $\dot{\rho}\colon H \to \mathrm{Aut}(\mathfrak{n})$ is the representation induced by ρ.*

Proof. γ is H-equivariant, i.e. for any $h \in H$, we have $\gamma \circ \mathrm{R}_h = \mathrm{R}_{(h,\mathrm{e}_N)} \circ \gamma$. Thus, the Ad-equivariance of $\hat{\boldsymbol{\omega}}$ implies

$$\begin{aligned} \mathrm{R}_h^*(\gamma^*\hat{\boldsymbol{\omega}}) &= (\gamma \circ \mathrm{R}_h)^*\hat{\boldsymbol{\omega}} \\ &= (\mathrm{R}_{(h,\mathrm{e}_N)} \circ \gamma)^*\hat{\boldsymbol{\omega}} \\ &= \gamma^*(\mathrm{R}_{(h,\mathrm{e}_N)}^*\hat{\boldsymbol{\omega}}) \\ &= \gamma^*(\mathrm{Ad}_{(h^{-1},\mathrm{e}_N)} \circ \hat{\boldsymbol{\omega}}) \\ &= \mathrm{Ad}_{(h^{-1},\mathrm{e}_N)} \circ (\gamma^*\hat{\boldsymbol{\omega}}). \end{aligned} \tag{4.4}$$

Considering the form of the adjoint representation for a semidirect product (proposition A.4), this means that

$$\mathrm{R}_h^*\boldsymbol{\omega} = \mathrm{Ad}_{h^{-1}} \circ \boldsymbol{\omega}\,, \tag{4.5a}$$

$$\mathrm{R}_h^*\boldsymbol{\theta} = \dot{\rho}_{h^{-1}} \circ \boldsymbol{\theta}\,. \tag{4.5b}$$

Now considering the fundamental vector fields of the actions on the principal bundles, the equivariance of γ implies that for $X \in \mathfrak{h}$, we have

$$\begin{aligned} \mathrm{D}\gamma(\tilde{X}(p)) &= \mathrm{D}\gamma\left(\frac{\mathrm{d}}{\mathrm{d}t} p\exp(tX)\Big|_{t=0}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\gamma(p\exp(tX))\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\gamma(p)\cdot(\exp(tX),\mathrm{e}_N)\Big|_{t=0} \\ &= \widetilde{(X,0)}(\gamma(p)), \end{aligned} \tag{4.6}$$

i.e. $\mathrm{D}\gamma \circ \tilde{X} = \widetilde{(X,0)} \circ \gamma$. Therefore, the condition $\hat{\boldsymbol{\omega}}\left(\widetilde{(X,Y)}(q)\right) = (X,Y)$ for $(X,Y) \in \mathfrak{h} \oplus \mathfrak{n}$ implies

$$(\gamma^*\hat{\boldsymbol{\omega}})(\tilde{X}(p)) = \hat{\boldsymbol{\omega}}(\mathrm{D}\gamma(\tilde{X}(p))) = \hat{\boldsymbol{\omega}}\left(\widetilde{(X,0)}(\gamma(p))\right) = (X,0), \tag{4.7}$$

i.e. ω satisfies

$$\omega(\tilde{X}) = X \tag{4.8}$$

and θ vanishes on vertical vectors. □

Lemma 4.3. *The above correspondence between connections on Q and pairs of connections and $\dot{\rho}$-tensorial one-forms on P is bijective, i.e. given a connection $\omega \in \Omega^1(P, \mathfrak{h})$ and a $\dot{\rho}$-tensorial $\theta \in \Omega^1_{\dot{\rho}}(P, \mathfrak{n})$, there is a unique connection $\hat{\omega} \in \Omega^1(Q, \mathfrak{h} \oplus \mathfrak{n})$ such that $\gamma^* \hat{\omega} = (\omega, \theta)$.*

Proof. Let $q \in Q$, and write it in the (unique) form $q = [p, (e_H, n)]$. For $v \in T_q Q$, we want to define $\hat{\omega}(v)$. Since $q = [p, (e_H, e_N)] \cdot (e_H, n) = \gamma(p) \cdot (e_H, n)$, we need

$$\begin{aligned}
\hat{\omega}_q(v) &= (\mathrm{R}^*_{(e_H,n)} \hat{\omega})_{\gamma(p)} \left(\mathrm{DR}_{(e_H, n^{-1})}(v) \right) \\
&= \mathrm{Ad}_{(e_H, n^{-1})} \left(\hat{\omega}_{\gamma(p)} \left(\mathrm{DR}_{(e_H, n^{-1})}(v) \right) \right).
\end{aligned} \tag{4.9}$$

Thus we need only consider the case $q = \gamma(p)$. So let $v \in T_{\gamma(p)} Q$. Such a v is not necessarily in the image of $\mathrm{D}\gamma$, but since the 'new directions' in TQ (new compared to TP) are 'along N' in the fibres, v can be uniquely written as

$$v = \mathrm{D}\gamma(\hat{v}) + \widetilde{(0, Y)}(\gamma(p)) \tag{4.10a}$$

with $\hat{v} \in T_p P$ and $Y \in \mathfrak{n}$. In the case $\hat{v} = 0$, we need $\hat{\omega}(v) = (0, Y)$, and in the case $Y = 0$, we need $\hat{\omega}_{\gamma(p)}(\mathrm{D}\gamma(\hat{v})) = (\gamma^* \hat{\omega})_p(\hat{v}) = (\omega_p(\hat{v}), \theta_p(\hat{v}))$. Thus $\hat{\omega}$ is required to be given by

$$\hat{\omega}_{\gamma(p)}(v) = (\omega_p(\hat{v}), \theta_p(\hat{v}) + Y) \tag{4.10b}$$

on the image of γ.

We now have to check that the $\hat{\omega}$ thus defined – via (4.10) on the image of γ and extended to all of Q via (4.9) – is really a connection. Ad-equivariance by N holds by construction. Ad-equivariance by H on the image of γ can be seen as follows: for $v = \mathrm{D}\gamma(\hat{v}) + \widetilde{(0, Y)}(\gamma(p)) \in T_{\gamma(p)} Q$, we have

$$\begin{aligned}
\mathrm{DR}_{(h, e_N)}(v) &= \mathrm{DR}_{(h, e_N)}(\mathrm{D}\gamma(\hat{v})) + \mathrm{DR}_{(h, e_N)} \left(\widetilde{(0, Y)}(\gamma(p)) \right) \\
&= \mathrm{D}\gamma(\mathrm{DR}_h(\hat{v})) + \widetilde{\mathrm{Ad}_{(h^{-1}, e_N)} \widetilde{(0, Y)}}(\gamma(p) \cdot (h, e_N)) \\
&= \mathrm{D}\gamma(\mathrm{DR}_h(\hat{v})) + \widetilde{\mathrm{Ad}_{(h^{-1}, e_N)} \widetilde{(0, Y)}}(\gamma(ph)) \\
&= \mathrm{D}\gamma(\mathrm{DR}_h(\hat{v})) + \widetilde{\widetilde{(0, \dot{\rho}_{h^{-1}}(Y))}}(\gamma(ph)),
\end{aligned} \tag{4.11}$$

where we used H-equivariance of γ and the general fact that for fundamental vector fields of right actions, we have $(\mathrm{R}_{g^{-1}})_* \tilde{X} = \widetilde{\mathrm{Ad}_g(X)}$ (exercise!), implying $\mathrm{DR}_g(\tilde{X}(p)) =$

$\widetilde{\mathrm{Ad}_{g^{-1}}(X)}(pg)$. Using this, we obtain

$$\begin{aligned}(\mathrm{R}^*_{(h,\mathrm{e}_N)}\hat{\omega})_{\gamma(p)}(v) &= \hat{\omega}_{\gamma(ph)}(\mathrm{DR}_{(h,\mathrm{e}_N)}(v)) \\ &= \big(\boldsymbol{\omega}_{ph}(\mathrm{DR}_h(\hat{v})), \boldsymbol{\theta}_{ph}(\mathrm{DR}_h(\hat{v})) + \dot{\rho}_{h^{-1}}(Y)\big) \\ &= \big(\mathrm{Ad}_{h^{-1}}(\boldsymbol{\omega}_p(\hat{v})), \dot{\rho}_{h^{-1}}(\boldsymbol{\theta}_p(\hat{v}) + Y)\big) \\ &= \mathrm{Ad}_{(h^{-1},\mathrm{e}_N)}(\hat{\omega}_{\gamma(p)}(v))\end{aligned} \tag{4.12}$$

where we used the Ad-equivariance of $\boldsymbol{\omega}$ and the $\dot{\rho}$-equivariance of $\boldsymbol{\theta}$. Combined, the Ad-equivariance of $\hat{\omega}$ on the whole of Q follows: for $q = \gamma(p)\cdot(\mathrm{e}_H, n)$, we have

$$\begin{aligned}(\mathrm{R}^*_{(h,\mathrm{e}_N)}\hat{\omega})_q &= \hat{\omega}_{q\cdot(h,\mathrm{e}_N)} \circ \mathrm{DR}_{(h,\mathrm{e}_N)} \\ &= \hat{\omega}_{\gamma(ph)\cdot(\mathrm{e}_H,\rho_{h^{-1}}(n))} \circ \mathrm{DR}_{(h,\mathrm{e}_N)} \\ \text{(4.9)}\quad &= \mathrm{Ad}_{(\mathrm{e}_H,\rho_{h^{-1}}(n^{-1}))} \circ \hat{\omega}_{\gamma(ph)} \circ \underbrace{\mathrm{DR}_{(\mathrm{e}_H,\rho_{h^{-1}}(n^{-1}))} \circ \mathrm{DR}_{(h,\mathrm{e}_N)}}_{=\mathrm{DR}_{(h,n^{-1})} = \mathrm{DR}_{(\mathrm{e}_H,n^{-1})\cdot(h,\mathrm{e}_N)}} \\ &= \mathrm{Ad}_{(\mathrm{e}_H,\rho_{h^{-1}}(n^{-1}))} \circ \hat{\omega}_{\gamma(ph)} \circ \mathrm{DR}_{(h,\mathrm{e}_N)} \circ \mathrm{DR}_{(\mathrm{e}_H,n^{-1})} \\ \text{(4.12)}\quad &= \mathrm{Ad}_{(\mathrm{e}_H,\rho_{h^{-1}}(n^{-1}))} \circ \mathrm{Ad}_{(h^{-1},\mathrm{e}_N)} \circ \hat{\omega}_{\gamma(p)} \circ \mathrm{DR}_{(\mathrm{e}_H,n^{-1})} \\ &= \mathrm{Ad}_{(h^{-1},\mathrm{e}_N)} \circ \mathrm{Ad}_{(\mathrm{e}_H,n^{-1})} \circ \hat{\omega}_{\gamma(p)} \circ \mathrm{DR}_{(\mathrm{e}_H,n^{-1})} \\ \text{(4.9)}\quad &= \mathrm{Ad}_{(h^{-1},\mathrm{e}_N)} \circ \hat{\omega}_q\ .\end{aligned} \tag{4.13}$$

Finally, we have to check that $\hat{\omega}$ take the correct value on fundamental vector fields. Since we already established Ad-equivariance, it is enough to show this on the image of γ. On fundamental vectors of the form $\widetilde{(0,Y)}(\gamma(p))$, our form $\hat{\omega}$ takes the value $(0,Y)$ by construction; and for $\widetilde{(X,0)}(\gamma(p))$, we finally have

$$\begin{aligned}\hat{\omega}\left(\widetilde{(X,0)}(\gamma(p))\right) &= \hat{\omega}\left(\frac{\mathrm{d}}{\mathrm{d}t}\gamma(p)\cdot\exp(t(X,0))\bigg|_{t=0}\right) \\ &= \hat{\omega}\left(\frac{\mathrm{d}}{\mathrm{d}t}\gamma(p)\cdot(\exp(tX),\mathrm{e}_N)\bigg|_{t=0}\right) \\ &= \hat{\omega}\left(\frac{\mathrm{d}}{\mathrm{d}t}\gamma(p\exp(tX))\bigg|_{t=0}\right) \\ &= \hat{\omega}(\mathrm{D}\gamma(\tilde{X}(p))) \\ \text{(4.10)}\quad &= (\boldsymbol{\omega}(\tilde{X}(p)), \boldsymbol{\theta}(\tilde{X}(p))) \\ &= (X,0).\end{aligned} \tag{4.14}$$

□

Theorem 4.4. *Let H, N be Lie groups, $\rho\colon H \to \mathsf{Aut}(N)$ a Lie group homomorphism, and $P \xrightarrow{\pi} M$ be a principal H-bundle. As in construction 4.1, let $\tilde{\rho}\colon H \to \mathsf{Diff}(H \ltimes N)$ be the natural left multiplication action of H on $H \ltimes N$, let $Q = P \times_{\tilde{\rho}} (H \ltimes N)$ the 'semidirect extension' of P via ρ, and $\gamma\colon P \to Q$ the natural embedding.*

Then connections $\hat{\boldsymbol{\omega}} \in \Omega^1(Q, \mathfrak{h} \oplus \mathfrak{n})$ on Q correspond bijectively to pairs of connections $\boldsymbol{\omega} \in \Omega^1(P, \mathfrak{h})$ and $\dot{\rho}$-tensorial one-forms $\boldsymbol{\theta} \in \Omega^1_{\dot{\rho}}(P, \mathfrak{n})$ on P via the pullback condition

$$\gamma^* \hat{\boldsymbol{\omega}} = (\boldsymbol{\omega}, \boldsymbol{\theta}), \tag{4.15}$$

where $\dot{\rho}\colon H \to \mathsf{Aut}(\mathfrak{n})$ is the representation induced by ρ.

In this situation, the curvature form $\hat{\boldsymbol{R}} \in \Omega^2(Q, \mathfrak{h} \oplus \mathfrak{n})$ of $\hat{\boldsymbol{\omega}}$ satisfies

$$\gamma^* \hat{\boldsymbol{R}} = (\boldsymbol{R}, \mathrm{d}^{\omega}\boldsymbol{\theta} + \tfrac{1}{2}[\boldsymbol{\theta} \wedge \boldsymbol{\theta}]), \tag{4.16}$$

where $\boldsymbol{R} \in \Omega^2(p, \mathfrak{h})$ is the curvature form of $\boldsymbol{\omega}$.

Proof. The bijective correspondence was established in lemmas 4.2 and 4.3. Pulling back the structure equation $\hat{\boldsymbol{R}} = \mathrm{d}\hat{\boldsymbol{\omega}} + \frac{1}{2}[\hat{\boldsymbol{\omega}} \wedge \hat{\boldsymbol{\omega}}]$ with γ and using the explicit form of the semidirect sum Lie bracket, we obtain

$$\begin{aligned} \gamma^* \hat{\boldsymbol{R}} &= \mathrm{d}(\gamma^* \hat{\boldsymbol{\omega}}) + \tfrac{1}{2}[\gamma^* \hat{\boldsymbol{\omega}} \wedge \gamma^* \hat{\boldsymbol{\omega}}] \\ &= \mathrm{d}(\boldsymbol{\omega}, \boldsymbol{\theta}) + \tfrac{1}{2}[(\boldsymbol{\omega}, \boldsymbol{\theta}) \wedge (\boldsymbol{\omega}, \boldsymbol{\theta})] \\ &= (\mathrm{d}\boldsymbol{\omega}, \mathrm{d}\boldsymbol{\theta}) + \tfrac{1}{2}([\boldsymbol{\omega} \wedge \boldsymbol{\omega}], [\boldsymbol{\theta} \wedge \boldsymbol{\theta}] + 2\dot{\rho}'_{\omega} \wedge \boldsymbol{\theta}) \\ &= (\boldsymbol{R}, \mathrm{d}^{\omega}\boldsymbol{\theta} + \tfrac{1}{2}[\boldsymbol{\theta} \wedge \boldsymbol{\theta}]). \end{aligned} \tag{4.17}$$

□

We can now apply this general theory to the classical example of the *affine frame bundle* of a manifold.

Example 4.5. We consider the above situation in the case that $P = F(M)$ is the linear frame bundle of an n-dimensional manifold, i.e. $H = \mathsf{GL}(n)$, and we let $N = \mathbb{R}^n$ and $\rho\colon \mathsf{GL}(n) \to \mathsf{Aut}(\mathbb{R}^n)$ the defining representation; i.e. $H \ltimes N = \mathsf{GL}(n) \ltimes \mathbb{R}^n$ is the *affine group* in n dimensions. The extended bundle $Q = F(M) \times_{\mathsf{GL}(n)} (\mathsf{GL}(n) \ltimes \mathbb{R}^n)$ from above is then naturally isomorphic to the *affine frame bundle* $\mathrm{Aff}(M)$ of M, whose fibres are given by

$$\mathrm{Aff}_x(M) = F_x(M) \times T_x M, \tag{4.18}$$

i.e. whose elements are 'affine bases' of tangent spaces. The isomorphism is given by

$$Q \ni [(\mathrm{e}_A), (\mathbb{1}, (y^A))] \mapsto ((\mathrm{e}_A), y^B \mathrm{e}_B) \in \mathrm{Aff}(M) \tag{4.19}$$

A connection on $\mathrm{Aff}_x(M)$, or equivalently one on Q, is called a *generalised affine connection* on M. According to theorem 4.4, such a connection $\boldsymbol{\omega}$ consists of a usual linear connection $\boldsymbol{\omega}$ on $F(M)$ and a tensorial form $\boldsymbol{\theta} \in \Omega^1_{\mathsf{GL}(n)}(F(M), \mathbb{R}^n)$.

The tensorial form $\boldsymbol{\theta}$ corresponds to a form on the base manifold valued in an associated vector bundle,

$$\theta \in \Omega^1(M, F(M) \times_{\mathrm{GL}(n)} \mathbb{R}^n), \tag{4.20a}$$

locally defined by

$$\theta = [\sigma, \sigma^* \boldsymbol{\theta}] \tag{4.20b}$$

for local sections σ. If this θ is the canonical solder form of $F(M) \times_{\mathrm{GL}(n)} \mathbb{R}^n$, then the generalised affine connection we started with is called an *affine connection* on $\mathrm{Aff}(M)$.

For a generalised affine connection given by $(\boldsymbol{\omega}, \boldsymbol{\theta})$, according to theorem 4.4 the 'translational part' of its curvature is given by the exterior covariant derivative $\mathrm{d}^{\boldsymbol{\omega}} \boldsymbol{\theta} \in \Omega^2_{\mathrm{GL}(n)}(F(M), \mathbb{R}^n)$, which corresponds to a form $\mathrm{d}^{\omega} \theta \in \Omega^2(M, F(M) \times_{\mathrm{GL}(n)} \mathbb{R}^n)$ on the base. In the case of an *affine* connection, i.e. if θ is the canonical solder form, according to Cartan's first structure equation (proposition 3.14 (ii)) this is the torsion of $\boldsymbol{\omega}$ (up to the identification $F(M) \times_{\mathrm{GL}(n)} \mathbb{R}^n \cong TM$ via θ).

5. Bargmann structures

In this chapter, we will use the machinery of 'semidirect extensions' developed in the previous chapter to describe the geometry of Galilei manifolds in terms of the so-called Bargmann group. We will see how this allows for a description of the coupling of massive matter to Newton–Cartan gravity via variational principles, and how this allows for a computationally easier description of the limit from GR to Newton–Cartan gravity than that we encountered before.

5.1. The Bargmann group

Definition 5.1. The *Bargmann group* in $n+1$ dimensions is the semidirect product

$$\mathsf{Barg} = \mathsf{Gal} \ltimes_\rho (\mathbb{R}^{n+1} \times \mathbb{R}), \tag{5.1a}$$

where the homomorphism $\rho\colon \mathsf{Gal} \to \mathsf{Aut}(\mathbb{R}^{n+1} \times \mathbb{R})$ is given by

$$\rho_{(R,k)}(y^A, \varphi) = \left(y^t, R^a{}_b y^b + y^t k^a, \varphi + \frac{1}{2}|k|^2 y^t + k_a R^a{}_b y^b\right). \tag{5.1b}$$

From (5.1b), we directly obtain the induced homomorphisms $\dot\rho\colon \mathsf{Gal} \to \mathsf{Aut}(\mathbb{R}^{n+1} \oplus \mathbb{R})$ and $\dot\rho'\colon \mathfrak{gal} \to \mathsf{Der}(\mathbb{R}^{n+1} \oplus \mathbb{R})$ as

$$\dot\rho_{(R,k)}(y^A, \varphi) = \left(y^t, R^a{}_b y^b + y^t k^a, \varphi + \tfrac{1}{2}|k|^2 y^t + k_a R^a{}_b y^b\right), \tag{5.2}$$

$$\dot\rho'_{(X,k)}(y^A, \varphi) = \left(((X,k)y)^A, k_a y^a\right) \tag{5.3}$$

(check this as an exercise!).

The Bargmann algebra $\mathfrak{barg} = \mathfrak{gal} \oplus (\mathbb{R}^{n+1} \oplus \mathbb{R})$ is a one-dimensional central extension of the inhomogeneous Galilei algebra $\mathfrak{igal} = \mathfrak{gal} \oplus \mathbb{R}^{n+1}$, i.e. we have a short exact sequence

$$0 \to \mathbb{R} \to \mathfrak{barg} \to \mathfrak{igal} \to 0 \tag{5.4}$$

of Lie algebras where the image of $\mathbb{R}$ in $\mathfrak{barg}$ lies in the centre (i.e. commutes with all elements). In fact, up to isomorphism and choice of a constant prefactor in the second component of 5.3, the Bargmann algebra is, for $n \neq 2$, essentially the *unique* one-dimensional non-trivial central extension of the inhomogeneous Galilei algebra.[1]

5.2. Extending Galilei to Bargmann structures

Construction 5.2. Let (M, τ, h) be a Galilei manifold, and $G(M)$ its Galilei frame bundle. Using construction 4.1, we extend $G(M)$ to a principal Barg-bundle $B(M) = G(M) \times_{\mathsf{Gal}} \mathsf{Barg}$. According to theorem 4.4, connections $\hat{\boldsymbol{\omega}}$ on $B(M)$ are in one-to-one correspondence with pairs $(\boldsymbol{\omega}, \boldsymbol{\Theta})$ of connections $\boldsymbol{\omega}$ and $\dot{\rho}$-tensorial one-forms $\boldsymbol{\Theta} \in \Omega^1_{\dot{\rho}}(G(M), \mathbb{R}^{n+1} \oplus \mathbb{R})$ on $G(M)$ via the pullback condition

$$\gamma^* \hat{\boldsymbol{\omega}} = (\boldsymbol{\omega}, \boldsymbol{\Theta}), \tag{5.5a}$$

where $\gamma \colon G(M) \to B(M)$ is the natural embedding and $\dot{\rho} \colon \mathsf{Gal} \to \mathsf{Aut}(\mathbb{R}^{n+1} \oplus \mathbb{R})$ is the induced representation (5.2). Furthermore, in this situation the pullback of the curvature form $\hat{\boldsymbol{R}}$ of $\hat{\boldsymbol{\omega}}$ is given by the curvature form $\boldsymbol{R}$ of $\boldsymbol{\omega}$ and the exterior covariant derivative of $\boldsymbol{\Theta}$ with respect to $\boldsymbol{\omega}$,

$$\gamma^* \hat{\boldsymbol{R}} = (\boldsymbol{R}, \mathrm{d}^{\boldsymbol{\omega}} \boldsymbol{\Theta}). \tag{5.5b}$$

We further decompose

$$\boldsymbol{\Theta} = (\boldsymbol{\theta}, \boldsymbol{a}) \tag{5.6}$$

with $\boldsymbol{\theta} \in \Omega^1(G(M), \mathbb{R}^{n+1})$ and $\boldsymbol{a} \in \Omega^1(G(M))$. Due to $\mathbb{R} \subset \mathbb{R}^{n+1} \oplus \mathbb{R}$ being a $\dot{\rho}$-invariant subspace, we may view the $\mathbb{R}^{n+1}$-valued part $\boldsymbol{\theta}$ as transforming under the quotient representation $\mathsf{Gal} \to \mathrm{GL}((\mathbb{R}^{n+1} \oplus \mathbb{R})/\mathbb{R}) \cong \mathrm{GL}(n+1)$, which is the usual representation of Gal on $\mathbb{R}^{n+1}$ (as in remark 3.3). This means that $\boldsymbol{\theta}$ is by itself a tensorial form

$$\boldsymbol{\theta} \in \Omega^1_{\mathsf{Gal}}(G(M), \mathbb{R}^{n+1}), \tag{5.7}$$

which naturally corresponds to an associated-bundle-valued form

$$\theta \in \Omega^1(M, G(M) \times_{\mathsf{Gal}} \mathbb{R}^{n+1}) \tag{5.8}$$

on our Galilei manifold. If this is the canonical solder form of $G(M) \times_{\mathsf{Gal}} \mathbb{R}^{n+1}$, then we call $\boldsymbol{a} \in \Omega^1(G(M))$ a *Bargmann structure*[2] on (M, τ, h).

[1] In $n = 2$ spatial dimensions, the space of 1d central extensions of the inhomogeneous Galilei algebra is three-dimensional instead of one-dimensional.

[2] Usually, given a Lie group G, by a G *structure* on a manifold M one means a reduction of the structure group of the linear frame bundle $F(M)$ from $\mathrm{GL}(\dim M)$ to G. However, in the case of the Bargmann group no misunderstandings can arise, since the Bargmann group in $n+1$ dimensions is not naturally a subgroup of $\mathrm{GL}(n+1)$.

Summed up, we have the following:

Definition 5.3. A *Bargmann structure* on a Galilei manifold (M, τ, h) is a one-form $\boldsymbol{a} \in \Omega^1(G(M))$ on the Galilei frame bundle that together with the tensorial form $\boldsymbol{\theta} \in \Omega^1_{\mathrm{Gal}}(G(M), \mathbb{R}^{n+1})$ corresponding to the canonical solder form of $G(M) \times_{\mathrm{Gal}} \mathbb{R}^{n+1}$ combines into a $\dot{\rho}$-tensorial form $(\boldsymbol{\theta}, \boldsymbol{a}) \in \Omega^1_{\dot{\rho}}(G(M), \mathbb{R}^{n+1} \oplus \mathbb{R})$.

In turn, together with a Galilei connection $\boldsymbol{\omega} \in \Omega^1(G(M), \mathfrak{gal})$, this $(\boldsymbol{\theta}, \boldsymbol{a})$ would give a 'Bargmann connection' $\hat{\boldsymbol{\omega}}$ on $B(M)$. Note however that we consider the choice of Galilei connection $\boldsymbol{\omega}$ *not* to be part of the choice of Bargmann structure.

Construction 5.4. Given a Bargmann structure $\boldsymbol{a}$ on a Galilei manifold (M, τ, h), the tensorial form $\boldsymbol{\Theta} = (\boldsymbol{\theta}, \boldsymbol{a}) \in \Omega^1_{\dot{\rho}}(G(M), \mathbb{R}^{n+1} \oplus \mathbb{R})$ corresponds to an associated-bundle-valued form $\Theta \in \Omega^1(M, G(M) \times_{\dot{\rho}} (\mathbb{R}^{n+1} \oplus \mathbb{R}))$. The local representative of this form with respect to a local Galilei frame $\sigma = (\mathrm{e}_A)$ defined on an open set $U \subset M$, i.e. the pullback

$$(\tau, \mathrm{e}^a, a) = \sigma^* \boldsymbol{\Theta} \in \Omega^1(U, \mathbb{R}^{n+1} \oplus \mathbb{R}) \tag{5.9a}$$

along the frame, using which we can locally express Θ as

$$\Theta = [\sigma, (\tau, \mathrm{e}^a, a)] \in \Omega^1\left(U, G(M) \times_{\dot{\rho}} (\mathbb{R}^{n+1} \oplus \mathbb{R})\right), \tag{5.9b}$$

we call an *extended coframe* following [GPR15]. (Note that since $\boldsymbol{\theta}$ corresponds to the canonical solder form, its local representative with respect to the local frame $\sigma = (\mathrm{e}_A) = (v, \mathrm{e}_a)$ is given by the dual frame (τ, e^a).)

We now consider the exterior covariant derivative $\mathrm{d}^{\boldsymbol{\omega}}\Theta$ of the form Θ, locally represented by the extended coframe, with respect to a Galilei connection $\boldsymbol{\omega}$. According to (5.5b), it corresponds to the $\mathbb{R}^{n+1} \oplus \mathbb{R}$-valued part of the curvature of the 'Bargmann connection' $\hat{\boldsymbol{\omega}}$ on $B(M)$ given by $\boldsymbol{\omega}$ and $\boldsymbol{\Theta}$. Since the $\mathbb{R}^{n+1}$-valued part of $\boldsymbol{\Theta}$ corresponds to the canonical solder form, according to Cartan's first structure equation (proposition 3.14 (ii)) the $\mathbb{R}^{n+1}$-valued part of $\mathrm{d}^{\boldsymbol{\omega}}\boldsymbol{\Theta}$ corresponds to the torsion of $\boldsymbol{\omega}$. Following [GPR15], we will therefore call $\mathrm{d}^{\boldsymbol{\omega}}\Theta$ the connection's *extended torsion* with respect to the Bargmann structure, and denote its local components with respect to a local frame $\sigma = (\mathrm{e}_A)$ by

$$(T^A, f) := \sigma^*(\mathrm{d}^{\boldsymbol{\omega}}\boldsymbol{\Theta}). \tag{5.10}$$

Its $\mathbb{R}$-valued part f, which of course does not define an invariant geometric object on its own, we call the *mass torsion*.[3]

[3]This name derives from the central direction of the Bargmann algebra being related to the mass of massive 'Galilei-invariant' (in fact locally $\mathrm{Gal} \times \mathrm{U}(1)$-invariant, as we will see later) matter fields.

Definition 5.5. Let (M, τ, h) be a Galilei manifold and $\boldsymbol{a} \in \Omega^1(G(M))$ a Bargmann structure on it. The *extended coframe* with respect to a local Galilei frame $\sigma = (\mathrm{e}_A) \in \Gamma(U, G(M))$ is $(\tau, \mathrm{e}^a, a) \in \Omega^1(U, \mathbb{R}^{n+1} \oplus \mathbb{R})$ with $a = \sigma^* \boldsymbol{a}$.

For a Galilei connection $\boldsymbol{\omega}$, its *extended torsion* with respect to the Bargmann structure $\boldsymbol{a}$ is the form $\mathrm{d}^\omega \Theta \in \Omega^2(M, G(M) \times_{\dot\rho} (\mathbb{R}^{n+1} \oplus \mathbb{R}))$, where Θ corresponds to the form $\boldsymbol{\Theta} = (\boldsymbol{\theta}, \boldsymbol{a}) \in \Omega^1_{\dot\rho}(G(M), \mathbb{R}^{n+1} \oplus \mathbb{R})$ with $\boldsymbol{\theta}$ corresponding to the canonical solder form. In the local representative $(T^A, f) := \sigma^*(\mathrm{d}^\omega \boldsymbol{\Theta})$, the $\mathbb{R}$ part f is called the *mass torsion*.

Proposition 5.6. *Let (M, τ, h) be a Galilei manifold and $\boldsymbol{a} \in \Omega^1(G(M))$ a Bargmann structure on it.*

(i) Under local rotations of Galilei frames, the $\mathbb{R}$ part of the extended coframe and the mass torsion f are invariant.

Under local Galilei boosts $(\mathrm{e}_A) \to (\tilde{\mathrm{e}}_A) = (\mathrm{e}_A) \cdot (\mathbb{1}, k)^{-1} = (v - k^a \mathrm{e}_a, \mathrm{e}_a)$ they transform as

$$a \to a + k_a \mathrm{e}^a + \frac{1}{2}|k|^2 \tau, \tag{5.11a}$$

$$f \to f + k_a T^a + \frac{1}{2}|k|^2 \mathrm{d}\tau. \tag{5.11b}$$

(ii) The mass torsion is explicitly given by

$$f = \mathrm{d}a + \varpi_a \wedge \mathrm{e}^a. \tag{5.12}$$

Proof. (i) The extended coframe transforms according to $\dot\rho$; i.e. for a general change of frame we have the transformation behaviour $(\tau, \mathrm{e}^a, a) \to (\tau, \tilde{\mathrm{e}}^a, \tilde{a}) = ((\mathrm{e}_A) \cdot (R, k)^{-1})^* \boldsymbol{\Theta} = \dot\rho_{(R,k)}((\mathrm{e}_A)^* \boldsymbol{\Theta})$. Using the explicit form (5.2) of $\dot\rho$, for $k = 0$ we obtain $\tilde{a} = a$; and for $R = \mathbb{1}$ we obtain the behaviour in the statement of the proposition.

The same applies for the extended torsion (T^A, f).

(ii) Using the explicit form (5.3) of $\dot\rho'$, we obtain

$$\begin{aligned}(T^A, f) &= \sigma^*(\mathrm{d}^\omega \boldsymbol{\Theta}) \\ &= \mathrm{d}(\mathrm{e}^A, a) + \dot\rho'_{(\omega,\varpi)} \wedge (\mathrm{e}^A, a) \\ &= (\mathrm{d}\mathrm{e}^A + \omega^A{}_B \wedge \mathrm{e}^B, \mathrm{d}a + \varpi_a \wedge \mathrm{e}^a),\end{aligned} \tag{5.13}$$

as stated.

□

Corollary 5.7. *On a Galilei manifold with Bargmann structure, the Newton–Coriolis form of a Galilei connection may be expressed in terms of the extended coframe and the mass torsion as*

$$\Omega = f - \mathrm{d}a. \tag{5.14}$$

Proof. This follows directly by combining proposition 5.6 (ii) with proposition 3.14 (i) (which stated $\Omega = \varpi_a \wedge \mathrm{e}^a$). □

Theorem 5.8. *On a Galilei manifold with a chosen Bargmann structure, Galilei connections are uniquely determined by their extended torsion.*

In particular, if the Galilei manifold has absolute time, there is a unique Galilei connection with vanishing extended torsion.[4] *This connection is Newtonian.*

Proof. Since according to the classification theorem (theorem 1.16) Galilei connections are determined by their torsion and Newton–Coriolis form, this follows directly from corollary 5.7.

In the case of vanishing extended torsion, the connection being Newtonian follows by $\mathrm{d}\Omega = -\mathrm{d}\mathrm{d}a = 0$ (using theorem 1.27). □

Remark 5.9. If on a Newtonian manifold (M, τ, h, ∇) we have a globally defined unit timelike vector field v such that the Newton–Coriolis form of ∇ with respect to v is exact, i.e. we can write $\Omega = -\mathrm{d}a$, then this a gives rise to a Bargmann structure $\boldsymbol{a}$ whose corresponding extended-torsion-free Galilei connection is the original Newtonian connection ∇.

In fact, a may be lifted to $G(M)$ to give the desired Bargmann structure $\boldsymbol{a}$: even though Galilei frames need not globally exist, around any point in M we may choose a Galilei frame whose timelike vector field is v; and since the local representative of a Bargmann structure is invariant under local frame rotations (i.e. local Galilei transformations that leave the timelike vector invariant) according to proposition 5.6 (i), our globally defined a has the correct transformation behaviour. The mass torsion of ∇ with respect to $\boldsymbol{a}$ and v vanishes by construction, so ∇ has vanishing extended torsion.

If Ω is not exact, then by the Poincaré lemma we still get local forms a such that $\Omega = -\mathrm{d}a$ (since Ω is closed by ∇ being Newtonian), but those don't 'patch up' to a global Bargmann structure.

Note that when obtaining a Newtonian manifold as a formal $c \to \infty$ limit of a Lorentzian manifold as in theorem 2.35, according to remark 2.36 the Newton–Coriolis form of the limiting connection ∇ with respect to the unit timelike vector field v

[4] Note that the vanishing of the extended torsion is a statement that makes sense independently of the choice of Galilei frame. This follows either by the transformation behaviour (5.11b), or by noting that vanishing extended torsion just means $\mathrm{d}^\omega \Theta = 0$, which is a global statement.

that naturally arises from the formal power series expansion in c^{-1} is of the form $\Omega = \tau \wedge \mathrm{d}\phi = -\mathrm{d}(\phi\tau)$, so we obtain a Bargmann structure with local representative $a = \phi\tau$.

The same is true for Newtonian spacetimes with absolute rotation that arise from geometrisation of solutions of standard Newtonian gravity, i.e. by performing the converse construction of the Trautman recovery theorem (theorem 2.29) as in remark 2.23: the constructed Newton–Coriolis form is again $\Omega = \tau \wedge \mathrm{d}\phi = -\mathrm{d}(\phi\tau)$, so we get a Bargmann structure with $a = \phi\tau$.

Remark 5.10. On a Galilei manifold (M, τ, h) we have an $\mathbb{R}$ gauge freedom for the Bargmann structure, corresponding to the $\mathbb{R}$ direction in the Bargmann group: given any $\mathbb{R}$-valued function χ on M, we obtain a gauge transformation f_χ of the principal Barg-bundle $B(M) \xrightarrow{\hat{\pi}} M$, i.e. a fibre-preserving principal bundle automorphism $f_\chi \colon B(M) \to B(M)$, as

$$f_\chi(p) := p \cdot (\mathbb{1}, 0, 0, \chi \circ \hat{\pi}). \tag{5.15a}$$

Given a connection $\hat{\omega}$ on $B(M)$, we may act on it (actively!) with the gauge transformation f_χ, giving the new connection

$$f_\chi^* \hat{\omega} = \hat{\omega} + (0, 0, 0, \hat{\pi}^*(\mathrm{d}\chi)) \tag{5.15b}$$

For the case of $\hat{\omega}$ given by a Galilei connection $\boldsymbol{\omega}$ and a Bargmann structure $\boldsymbol{a}$ on (M, τ, h), this means that applying the gauge transformation given by χ, we obtain a new Bargmann structure

$$\boldsymbol{a} \to \boldsymbol{a} + \pi^*(\mathrm{d}\chi), \tag{5.15c}$$

which on M is locally represented by

$$a \to a + \mathrm{d}\chi. \tag{5.15d}$$

Note that with respect to two Bargmann structures which are related to each other by such an $\mathbb{R}$ gauge transformation, a Galilei connection has the same extended torsion, since $f = \mathrm{d}a + \Omega = \mathrm{d}(a + \mathrm{d}\chi) + \Omega$.

In particular, the Bargmann structures constructed in remark 5.9 from Newtonian connections with globally exact Newton–Coriolis forms are unique up to such a gauge transformation.

5.3. Variational description of coupling to massive matter

Construction 5.11 (The massive point particle)**.** Let (M, τ, h) be a Galilei manifold with a Bargmann structure $\boldsymbol{a}$. We consider the following action functional for a timelike worldline γ:

$$\begin{aligned} S[\gamma] &= \int_{\lambda_i}^{\lambda_f} \mathrm{d}\lambda \left(\frac{1}{2} m \frac{\underset{v}{h}(\gamma', \gamma')}{\tau(\gamma')} - m a(\gamma') \right) \\ &= \int_\gamma \left(\frac{1}{2} m \frac{\underset{v}{h}(\cdot, \cdot)}{\tau} - m a \right) \end{aligned} \tag{5.16}$$

Note that the first summand in the integrand in fact defines a one-form along the curve, since for any $\mu \in \mathbb{R}$ we have $\frac{\underset{v}{h}(\mu \cdot \gamma', \mu \cdot \gamma')}{\tau(\mu \cdot \gamma')} = \mu \cdot \frac{\underset{v}{h}(\gamma', \gamma')}{\tau(\gamma')}$. Since the action is an integral over a one-form, it is reparametrisation invariant, i.e. depends on γ only through its image.

The expression (5.16) for the action explicitly depends on the choice of Galilei frame: both the covariant space metric $\underset{v}{h}$ and the local representative a of the Bargmann structure depend on the choice of unit timelike vector field v. In fact, however, the action is independent of the choice of frame: under a local Galilei boost (Milne boost) $v \to v - k^a e_a$, $\underset{v}{h}$ and a transform as

$$\underset{v}{h} \to \underset{v}{h} + (k_a e^a) \otimes \tau + \tau \otimes (k_a e^a) + |k|^2 \tau \otimes \tau, \tag{5.17a}$$

$$a \to a + k_a e^a + \frac{1}{2} |k|^2 \tau \tag{5.17b}$$

according to proposition 1.19 and proposition 5.6 (i); therefore, the integrand transforms as

$$\begin{aligned} \frac{1}{2} m \frac{\underset{v}{h}(\gamma', \gamma')}{\tau(\gamma')} - m a(\gamma') \to{}& \frac{1}{2} m \left(\frac{\underset{v}{h}(\gamma', \gamma')}{\tau(\gamma')} + \cancel{2 k_a e^a(\gamma')} + \cancel{|k|^2 \tau(\gamma')} \right) \\ & - m \left(a(\gamma') + \cancel{k_a e^a(\gamma')} + \cancel{\frac{1}{2} |k|^2 \tau(\gamma')} \right) \\ ={}& \frac{1}{2} m \frac{\underset{v}{h}(\gamma', \gamma')}{\tau(\gamma')} - m a(\gamma'). \end{aligned} \tag{5.17c}$$

In the case of a Newtonian spacetime with absolute rotation that arose from a solution of the standard formulation of Newtonian gravity with $a = \phi \tau$ (as in remark 5.9), the action is just the usual action of a point particle in Newtonian mechanics, namely the integral over kinetic minus gravitational potential energy, $S = \int \mathrm{d}t (\frac{1}{2} m \delta_{ab} \dot{\gamma}^a \dot{\gamma}^b - m\phi)$. Thus one might say that the action (5.16) arises from the action of a particle in Newtonian mechanics by 'geometrisation'.

Note that the action (5.16) is not invariant under $\mathbb{R}$ gauge transformations of the Bargmann structure as in remark 5.10. However, under a gauge transformation by χ, it transforms as $S[\gamma] \to S[\gamma] - m \int_\gamma \mathrm{d}\chi = S[\gamma] - m(\chi(\gamma_f) - \chi(\gamma_i))$, i.e. by a boundary term.

Now we want to determine the equations of motion / Euler–Lagrange equations that arise from the above action by demanding it to be stationary under variations of γ with fixed endpoints. Computing the variation of the action, we first obtain

$$\begin{aligned}\delta S = \int \mathrm{d}\lambda \bigg[&\frac{1}{2} m \frac{(\delta h_{\mu\nu})\gamma'^\mu\gamma'^\nu}{\tau(\gamma')} + m\frac{h_{\mu\nu}\gamma'^\mu\delta\gamma'^\nu}{\tau(\gamma')} - m(\delta a_\mu)\gamma'^\mu - m a_\mu \delta\gamma'^\mu \\ &- \frac{1}{2} m \frac{\underset{v}{h}(\gamma',\gamma')}{(\tau(\gamma')^2)}\left((\delta\tau_\sigma)\gamma'^\sigma + \tau_\rho\delta\gamma'^\rho\right)\bigg] \\ = \int \mathrm{d}\lambda \bigg[&\frac{1}{2} m\delta\gamma^\rho \frac{(\partial_\rho h_{\mu\nu})\gamma'^\mu\gamma'^\nu}{\tau(\gamma')} - m\delta\gamma^\nu \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{h_{\mu\nu}\gamma'^\mu}{\tau(\gamma')}\right) \\ &\underbrace{- m\delta\gamma^\rho(\partial_\rho a_\mu)\gamma'^\mu + m\gamma'^\rho(\partial_\rho a_\mu)\delta\gamma^\mu}_{=m\delta\gamma^\rho(\mathrm{d}a)_{\mu\rho}\gamma'^\mu} - \frac{1}{2} m \frac{\underset{v}{h}(\gamma',\gamma')}{(\tau(\gamma')^2)}\delta\gamma^\rho(\partial_\rho\tau_\sigma)\gamma'^\sigma \\ &+ \frac{1}{2} m\delta\gamma^\rho \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\underset{v}{h}(\gamma',\gamma')}{(\tau(\gamma')^2)}\tau_\rho\right)\bigg], \end{aligned} \tag{5.18}$$

where we have used partial integration and vanishing of the variation $\delta\gamma^\mu$ on the boundary to express all terms involving the derivative $\delta\gamma'^\mu$ in terms of $\delta\gamma^\mu$ itself. Continuing the calculation, we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{h_{\mu\nu}\gamma'^\mu}{\tau(\gamma')}\right) = \frac{1}{\tau(\gamma')}(\partial_\rho h_{\mu\nu})\gamma'^\rho\gamma'^\mu + \frac{1}{\tau(\gamma')}h_{\mu\nu}\gamma''^\mu + h_{\mu\nu}\gamma'^\mu\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{1}{\tau(\gamma')}\right), \tag{5.19a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\underset{v}{h}(\gamma',\gamma')}{(\tau(\gamma')^2)}\tau_\rho\right) = \frac{\underset{v}{h}(\gamma',\gamma')}{(\tau(\gamma')^2)}(\partial_\sigma\tau_\rho)\gamma'^\sigma + \tau_\rho\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\underset{v}{h}(\gamma',\gamma')}{(\tau(\gamma')^2)}\right). \tag{5.19b}$$

For the respective last terms of these expressions we further obtain

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{1}{\tau(\gamma')}\right) = -\frac{1}{(\tau(\gamma'))^2}\frac{\mathrm{d}\tau(\gamma')}{\mathrm{d}\lambda}, \tag{5.19c}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\underset{v}{h}(\gamma',\gamma')}{(\tau(\gamma')^2)}\right) &= \frac{1}{(\tau(\gamma'))^2}\left((\partial_\kappa h_{\mu\nu})\gamma'^\kappa\gamma'^\mu\gamma'^\nu + 2h_{\mu\nu}\gamma'^\mu\gamma''^\nu\right) \\ &\quad - 2\frac{1}{(\tau(\gamma'))^3}\frac{\mathrm{d}\tau(\gamma')}{\mathrm{d}\lambda}\underset{v}{h}(\gamma',\gamma').\end{aligned} \tag{5.19d}$$

Into these we could further insert

$$\frac{\mathrm{d}\tau(\gamma')}{\mathrm{d}\lambda} = (\partial_\sigma\tau_\mu)\gamma'^\sigma\gamma'^\mu + \tau_\mu\gamma''^\mu, \tag{5.20}$$

but will refrain from doing so since the corresponding terms will not contribute to our end result.

Inserting (5.19) into (5.18), the action variation in the general case is given by

$$\begin{aligned}\delta S = \int \mathrm{d}\lambda\, \delta\gamma^\rho \bigg[& - m\frac{1}{\tau(\gamma')} h_{\mu\rho}\gamma''^\mu - \frac{1}{2}m(2\partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu})\frac{\gamma'^\mu\gamma'^\nu}{\tau(\gamma')} + m(\mathrm{d}a)_{\mu\rho}\gamma'^\mu \\ & - \frac{1}{2}m(\mathrm{d}\tau)_{\rho\sigma}\gamma'^\sigma \frac{\underset{v}{h}(\gamma',\gamma')}{(\tau(\gamma')^2)} + m h_{\mu\rho}\gamma'^\mu \frac{1}{(\tau(\gamma'))^2}\frac{\mathrm{d}\tau(\gamma')}{\mathrm{d}\lambda} \\ & + \frac{1}{2}m\tau_\rho \bigg(\frac{1}{(\tau(\gamma'))^2} \left((\partial_\kappa h_{\mu\nu})\gamma'^\kappa\gamma'^\mu\gamma'^\nu + 2h_{\mu\nu}\gamma'^\mu\gamma''^\nu \right) \\ & \qquad - 2\frac{1}{(\tau(\gamma'))^3}\frac{\mathrm{d}\tau(\gamma')}{\mathrm{d}\lambda}\underset{v}{h}(\gamma',\gamma') \bigg) \bigg]. \end{aligned} \tag{5.21}$$

We now assume that γ is parametrised by time along γ as defined by τ, i.e. we have $\gamma(t)$ with $\tau(\dot\gamma(t)) = 1$. In particular this implies $\frac{\mathrm{d}\tau(\dot\gamma)}{\mathrm{d}t} = 0$. Therefore, the equation of motion that we may read off from (5.21) simplifies to

$$\begin{aligned} 0 = {} & h_{\mu\rho}\ddot\gamma^\mu + \frac{1}{2}(2\partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu})\dot\gamma^\mu\dot\gamma^\nu - (\mathrm{d}a)_{\mu\rho}\dot\gamma^\mu + \frac{1}{2}(\mathrm{d}\tau)_{\rho\sigma}\dot\gamma^\sigma \underset{v}{h}(\dot\gamma,\dot\gamma) \\ & - \frac{1}{2}\tau_\rho \left((\partial_\kappa h_{\mu\nu})\dot\gamma^\kappa\dot\gamma^\mu\dot\gamma^\nu + 2h_{\mu\nu}\dot\gamma^\mu\ddot\gamma^\nu \right). \end{aligned} \tag{5.22}$$

Raising the free index, we obtain

$$0 = P^\rho_\sigma \ddot\gamma^\sigma + \frac{1}{2}h^{\rho\sigma}(2\partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu})\dot\gamma^\mu\dot\gamma^\nu - (\mathrm{d}a)_\mu{}^\rho\dot\gamma^\mu + \frac{1}{2}(\mathrm{d}\tau)^\rho{}_\sigma\dot\gamma^\sigma\underset{v}{h}(\dot\gamma,\dot\gamma). \tag{5.23}$$

Let now ∇ be an arbitrary Galilei connection on (M,τ,h). Using the classification theorem (theorem 1.16), we obtain

$$P^\rho_\sigma \Gamma^\sigma_{\mu\nu}\dot\gamma^\mu\dot\gamma^\nu = \frac{1}{2}h^{\rho\sigma}(2\partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu})\dot\gamma^\mu\dot\gamma^\nu - T_{\mu\nu}{}^\rho\dot\gamma^\mu\dot\gamma^\nu + \Omega_\mu{}^\rho\dot\gamma^\mu. \tag{5.24}$$

Comparing this to the equation of motion (5.23) and using $\Omega = f - \mathrm{d}a$, we see that the equation of motion is equivalent to

$$0 = P^\rho_\sigma \underbrace{(\ddot\gamma^\sigma + \Gamma^\sigma_{\mu\nu}\dot\gamma^\mu\dot\gamma^\nu)}_{=(\nabla_{\dot\gamma}\dot\gamma)^\sigma} + T_{\mu\nu}{}^\rho\dot\gamma^\mu\dot\gamma^\nu - f_\mu{}^\rho\dot\gamma^\mu + \frac{1}{2}(\mathrm{d}\tau)^\rho{}_\sigma\dot\gamma^\sigma\underset{v}{h}(\dot\gamma,\dot\gamma). \tag{5.25}$$

Combined with $\tau(\nabla_{\dot\gamma}\dot\gamma) = \nabla_{\dot\gamma}(\tau(\dot\gamma)) = \nabla_{\dot\gamma}(1) = 0$, we thus see that for curves parametrised by time along them (as defined by τ), the equation of motion arising from the above action is

$$(\nabla_{\dot\gamma}\dot\gamma)^\rho = -T_{\mu\nu}{}^\rho\dot\gamma^\mu\dot\gamma^\nu + f_\mu{}^\rho\dot\gamma^\mu - \frac{1}{2}(\mathrm{d}\tau)^\rho{}_\sigma\dot\gamma^\sigma\underset{v}{h}(\dot\gamma,\dot\gamma). \tag{5.26}$$

In the case of an extended-torsion-free connection, we have thus obtained the geodesic equation – i.e. the equation of motion for a massive test particle in Newton–Cartan gravity – from a variational principle. Note that the use of the Bargmann structure was crucial for this. Morally speaking, the Bargmann structure allows us to speak about the Newtonian gravitational potential ϕ, but in proper geometric terms.

Note that even though by parametrising the curve by time along it we may write the action in the easier form

$$S[\gamma] = \int_{t_i}^{t_f} \mathrm{d}t \left(\frac{1}{2} m h_{\mu\nu} \dot{\gamma}^\mu \dot{\gamma}^\nu - m a_\mu \dot{\gamma}^\mu \right), \tag{5.27}$$

in the general case this is misleading for calculating the variation of the action, since for fixed endpoints of the curve the time along it may change under variation if $\mathrm{d}\tau \neq 0$.

Construction 5.12 (The Schrödinger field)**.** The standard free 'non-relativistic' Schrödinger equation with mass m is

$$\mathrm{i}\hbar\partial_t\Psi = -\frac{\hbar^2}{2m}\delta^{ab}\partial_a\partial_b\Psi \tag{5.28}$$

for a $\mathbb{C}$-valued function Ψ. It may be obtained by variation of the action functional

$$S_{\text{free}}[\Psi] = \int \mathrm{d}t\mathrm{d}^n x \left(\mathrm{i}\hbar m(\overline{\Psi}\,\partial_t\Psi - \overline{\partial_t\Psi}\,\Psi) - \hbar^2\delta^{ab}\,\overline{\partial_a\Psi}\,\partial_b\Psi \right). \tag{5.29}$$

Here, we understand the Schrödinger equation as a *classical* field equation.[5]

We now want to couple the Schrödinger field to Newtonian gravity, described geometrically by Newton–Cartan gravity. In order to do so, we will 'geometrise' the above action, similar to how the point particle action in construction 5.11 is a 'geometrised' version of the free point particle action. As a first step, we observe that the term $\delta^{ab}\,\overline{\partial_a\Psi}\,\partial_b\Psi$ can be translated onto a general Galilei manifold (M, τ, h) simply as $h^{\mu\nu}\,\overline{\partial_\mu\Psi}\,\partial_\nu\Psi$. For the time derivative $\partial_t\Psi$ however we need to introduce a time direction – i.e. we need to replace this somehow by a term related to $v^\mu\partial_\mu\Psi$, where v is a unit timelike vector field. This would then introduce a dependence on the choice of v, i.e. the action would not be geometric in the sense of just depending on the Galilei manifold itself.

This apparent problem may however be solved with inspiration coming from the fact that the free Schrödinger equation is invariant under Galilei boosts only if those are implemented in a non-trivial manner, including a specific phase factor arising from the Bargmann group.[6] This was realised by Duval and Künzle in a paper from 1984 [DK84]. Their solution, adapted to our notation, works as follows.

[5]Of course, one might quantise the classical Schrödinger field, which would yield the usual 'second quantised' description of an arbitrary number of non-interacting quantum particles.

[6]Details of this will be studied in an exercise.

We consider a Bargmann structure $\boldsymbol{a}$ on our Galilei manifold (M, τ, h), and its local representative a with respect to a chosen unit timelike vector field v. The $\mathfrak{u}(1)$-valued (local) one-form $\mathrm{i}\frac{m}{\hbar}a$ on M then defines a connection on the trivial principal $\mathsf{U}(1)$-bundle $M \times \mathsf{U}(1)$. Note that this connection depends on the choice of v – a local Galilei boost will lead to a different local form a, and therefore to a different $\mathsf{U}(1)$ connection! The Schrödinger field is now a section in the associated vector bundle $E = (M \times \mathsf{U}(1)) \times_{\mathsf{U}(1)} \mathbb{C}$ with respect to the defining representation of $\mathsf{U}(1)$ on $\mathbb{C}$. We have a $\mathsf{U}(1)$-covariant derivative operator D on E, induced by our connection, acting on sections $\Psi \in \Gamma(E)$ as

$$\mathrm{D}_\mu \Psi = \left(\partial_\mu + \mathrm{i}\frac{m}{\hbar}a_\mu\right)\Psi. \tag{5.30}$$

We now use *this* covariant derivative operator to 'minimally couple' the free Schrödinger action to Newton–Cartan gravity, leading to the action

$$S[\Psi] = \int \mathrm{vol}\left(\mathrm{i}\hbar m v^\mu(\overline{\Psi}\,\mathrm{D}_\mu\Psi - \overline{\mathrm{D}_\mu\Psi}\,\Psi) - \hbar^2 h^{\mu\nu}\,\overline{\mathrm{D}_\mu\Psi}\,\mathrm{D}_\nu\Psi\right), \tag{5.31}$$

where $\mathrm{vol} = \tau \wedge \mathrm{e}^1 \wedge \cdots \wedge \mathrm{e}^n$ is the natural volume form of (M, τ, h). Even though it is not obvious from the construction, this action is in fact *invariant* under local Galilei boosts (exercise!). By construction, it is also invariant under $\mathbb{R}$ gauge transformations of the Bargmann structure as in remark 5.10 when we let those act on $M \times \mathsf{U}(1)$, to which E is associated, as well, via the group homomorphism $\mathbb{R} \to \mathsf{U}(1), \chi \mapsto \mathrm{e}^{\mathrm{i}\chi}$.

The Schrödinger equation we obtain from variation of this action is

$$\mathrm{i}\hbar v^\mu \mathrm{D}_\mu\Psi + \frac{\mathrm{i}\hbar}{2}\frac{1}{f_{\mathrm{vol}}}(\partial_\mu(f_{\mathrm{vol}}v^\mu))\Psi = -\frac{\hbar^2}{2m}\frac{1}{f_{\mathrm{vol}}}\left(\partial_\mu + \mathrm{i}\frac{m}{\hbar}a_\mu\right)(f_{\mathrm{vol}}h^{\mu\nu}\mathrm{D}_\nu\Psi), \tag{5.32a}$$

where f_{vol} is the coordinate component of the volume form, $\mathrm{vol} = f_{\mathrm{vol}}\,\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^n$. For a Galilei manifold with absolute time, we can further rewrite this as

$$\mathrm{i}\hbar v^\mu \mathrm{D}_\mu\Psi + \frac{\mathrm{i}\hbar}{2}(\nabla_\mu v^\mu)\Psi = -\frac{\hbar^2}{2m}h^{\mu\nu}\mathrm{D}_\mu\mathrm{D}_\nu\Psi, \tag{5.32b}$$

where ∇ is *any* torsion-free Galilei connection and we have extended the action of D to E-valued one-forms in the natural way, i.e.

$$\mathrm{D}_\mu\eta_\nu = \left(\nabla_\mu + \mathrm{i}\frac{m}{\hbar}a_\mu\right)\eta_\nu. \tag{5.33}$$

Note that equation (5.32b) is indeed independent of the choice of (torsion-free) Galilei connection: $\nabla_\mu v^\mu$ is (proportional to) the expansion of v, which depends only on v and h (see proposition 2.8); and on the right-hand side, we project onto space and thus see only the spatial part $\overset{(n)}{\nabla}$ of the connection, which is the spatial Levi-Civita connection (corollary 1.8).

In the case of a Newtonian manifold that arose from a solution of the standard formulation of Newtonian gravity, with $a = \phi\tau$, in adapted coordinates we have $v = \frac{\partial}{\partial t}$ and $h^{\mu\nu} = \delta^\mu_a \delta^\nu_b \delta^{ab}$, giving the Schrödinger equation

$$i\hbar\partial_t \Psi = -\frac{\hbar^2}{2m}\delta^{ab}\partial_a\partial_b\Psi + m\phi\Psi \tag{5.34}$$

as expected.

5.4. Bargmann structures from formal expansions of general relativity

In this section, we will explain how Bargmann structures may be included into the formal limiting process from Lorentzian geometry and GR to the geometry of Galilei manifolds and Newton–Cartan gravity, as presented in section 2.3.

Theorem 5.13. *Let (M, g) be a Lorentzian manifold, and let $(E_A) = (E_0, E_a)$ be a local orthonormal frame with dual frame (E^A), such that the metric and inverse metric can be written as*

$$g = \eta_{AB} E^A \otimes E^B, \quad g^{-1} = \eta^{AB} E_A \otimes E_B, \tag{5.35}$$

where η_{AB} denotes the components of the Minkowski metric in Lorentzian coordinates, i.e. $(\eta_{AB}) = \mathrm{diag}(-1, 1, \ldots, 1)$. Assume that the frame and dual frame may be expanded as formal power series in c^{-1} as

$$E^0 = c\tau + c^{-1}a + O(c^{-3}), \qquad E^a = e^a + O(c^{-2}), \tag{5.36a}$$

$$E_0 = c^{-1}v + O(c^{-3}), \qquad E_a = e_a + O(c^{-2}) \tag{5.36b}$$

for some nowhere vanishing one-form $\tau \in \Omega^1(M)$.

(i) The expansions of the metric g and inverse metric g^{-1} satisfy the assumptions of lemma 2.33, such that we obtain a Galilei manifold (M, τ, h) as a formal $c \to \infty$ limit of our Lorentzian manifold.

(ii) (v, e_a) is a local Galilei frame for the limiting Galilei manifold (M, τ, h), with dual frame (τ, e^a).

(iii) *We consider a local Lorentz boost* $(\Lambda^A{}_B)$ *parametrised by the* $\mathbb{R}^n$*-valued boost velocity function* k *as*

$$\Lambda = \exp(\zeta^a K_a) \tag{5.37a}$$

with the rapidity

$$\zeta^a = \operatorname{artanh}(|k|/c)\frac{k^a}{|k|} \tag{5.37b}$$

and the boost generators

$$(K_a)^A{}_B = \delta^A_0 \eta_{aB} - \delta^A_a \eta_{0B}. \tag{5.37c}$$

Explicitly, this means that

$$\Lambda^0{}_0 = 1 + c^{-2}\frac{|k|^2}{2} + \mathrm{O}(c^{-4}), \tag{5.37d}$$

$$\Lambda^a{}_0 = c^{-1}k^a + \mathrm{O}(c^{-3}) = \delta^{ab}\Lambda^0{}_b\,, \tag{5.37e}$$

$$\Lambda^a{}_b = \delta^a_b + c^{-2}\frac{k^a k_b}{2} + \mathrm{O}(c^{-4}). \tag{5.37f}$$

Transforming the Lorentzian frame (E_A) *by* Λ *according to*

$$(\mathrm{E}_A) \to (\tilde{\mathrm{E}}_A) = \left(\mathrm{E}_B (\Lambda^{-1})^B{}_A\right), \tag{5.38}$$

and expanding the new frame analogously to (5.36), we obtain a local Galilei boost of the Galilei frame (v, e_a) *with boost velocity parameter* k*.*

Furthermore, the local one-forms a *that arise as the* c^{-1} *component of the timelike dual frame one-form* E^0 *transform as the local representatives of a Bargmann structure on* (M, τ, h) *would, thereby in fact defining a Bargmann structure* $\boldsymbol{a}$*.*

(iv) *Let* $\overset{\mathrm{L}}{\omega}$ *be a Lorentzian metric connection with a regular formal* $c \to \infty$ *limit (i.e. its coordinate components with respect to* c*-independent coordinates, or equivalently its local connection form with respect to the frame* (v, e_a)*, have regular limits). Then its local connection form with respect to* (E_A) *expands as*

$$\overset{\mathrm{L}}{\omega}{}^0{}_0 = 0, \tag{5.39a}$$

$$\overset{\mathrm{L}}{\omega}{}^a{}_0 = c^{-1}\varpi^a + \mathrm{O}(c^{-3}) = \delta^{ab}\overset{\mathrm{L}}{\omega}{}^0{}_b\,, \tag{5.39b}$$

$$\overset{\mathrm{L}}{\omega}{}^a{}_b = \omega^a{}_b + \mathrm{O}(c^{-2}), \tag{5.39c}$$

for local one-forms ${\omega^a}_b, \varpi^a$. *Under local rotations and boosts of the frame, the* $({\omega^a}_b, \varpi^a)$ *transform as the local connection form of a Galilei connection on* (M, τ, h) *would, thereby defining a Galilei connection* $\boldsymbol{\omega}$. *The torsion* $\overset{\mathrm{L}}{T}$ *of* $\overset{\mathrm{L}}{\boldsymbol{\omega}}$ *then expands as*

$$\overset{\mathrm{L}}{T}{}^0 = c\mathrm{d}\tau + c^{-1}f + \mathrm{O}(c^{-3}), \tag{5.40a}$$

$$\overset{\mathrm{L}}{T}{}^a = T^a + \mathrm{O}(c^{-2}) \tag{5.40b}$$

in terms of the extended torsion of $\boldsymbol{\omega}$ *with respect to the Bargmann structure* $\boldsymbol{a}$ *obtained from the expansion of the frame in part (iii).*[7]

Proof. (i) From the duality conditions

$$1 = \mathrm{E}^0(\mathrm{E}_0) = \tau(v) + \mathrm{O}(c^{-2}), \tag{5.41a}$$

$$0 = \mathrm{E}^0(\mathrm{E}_a) = c\tau(\mathrm{e}_a) + \mathrm{O}(c^{-1}), \tag{5.41b}$$

$$0 = \mathrm{E}^a(\mathrm{E}_0) = c^{-1}\mathrm{e}^a(v) + \mathrm{O}(c^{-3}), \tag{5.41c}$$

$$\delta^a_b = \mathrm{E}^a(\mathrm{E}_b) = \mathrm{e}^a(\mathrm{e}_b) + \mathrm{O}(c^{-2}), \tag{5.41d}$$

we obtain that (v, e^a) and (τ, e_a) are a local frame of vector fields and the corresponding dual frame of one-forms, respectively. Inserting the frame expansions into the metric and inverse metric expressed in terms of the frames, we further obtain

$$\begin{aligned} g &= \eta_{AB}\mathrm{E}^A \otimes \mathrm{E}^B = -\mathrm{E}^0 \otimes \mathrm{E}^0 + \delta_{ab}\mathrm{E}^a \otimes \mathrm{E}^b \\ &= -c^2\tau \otimes \tau - \tau \otimes a - a \otimes \tau + \delta_{ab}\mathrm{e}^a \otimes \mathrm{e}^b + \mathrm{O}(c^{-2}), \end{aligned} \tag{5.42a}$$

$$\begin{aligned} g^{-1} &= \eta^{AB}\mathrm{E}_A \otimes \mathrm{E}_B = -\mathrm{E}_0 \otimes \mathrm{E}_0 + \delta^{ab}\mathrm{E}_a \otimes \mathrm{E}_b \\ &= \delta^{ab}\mathrm{e}_a \otimes \mathrm{e}_b + \mathrm{O}(c^{-2}). \end{aligned} \tag{5.42b}$$

Thus, we see that the assumptions of lemma 2.33 are satisfied with the given τ and $h = \delta^{ab}\mathrm{e}_a \otimes \mathrm{e}_b$.

(ii) The above equation for h in terms of the frame (v, e_a) and the fact that $\tau(v) = 1$ show that this frame is a Galilei frame.

(iii) Transforming the frame and dual frame by a local boost Λ parametrised as in

[7] Note that in particular we recover one direction of theorem 2.35 (ii): if the Levi-Civita connection, with $\overset{\mathrm{L}}{T} = 0$, has a regular limit, then $\mathrm{d}\tau = 0$.

(5.37), we obtain

$$\begin{aligned}
c\tilde{\tau} + c^{-1}\tilde{a} + \mathrm{O}(c^{-3}) &= \tilde{\mathrm{E}}^0 = \Lambda^0{}_A \mathrm{E}^A \\
&= \Lambda^0{}_0 \mathrm{E}^0 + \Lambda^0{}_a \mathrm{E}^a \\
&= \left(1 + c^{-2}\frac{|k|^2}{2} + \mathrm{O}(c^{-4})\right)(c\tau + c^{-1}a + \mathrm{O}(c^{-3})) \\
&\quad + (c^{-1}k_a + \mathrm{O}(c^{-3}))(\mathrm{e}^a + \mathrm{O}(c^{-2})) \\
&= c\tau + c^{-1}\left(a + \frac{|k|^2}{2}\tau + k_a \mathrm{e}^a\right) + \mathrm{O}(c^{-3}), && (5.43a) \\
\tilde{\mathrm{e}}^a + \mathrm{O}(c^{-2}) &= \tilde{\mathrm{E}}^a = \Lambda^a{}_B \mathrm{E}^B \\
&= \Lambda^a{}_0 \mathrm{E}^0 + \Lambda^a{}_b \mathrm{E}^b \\
&= (c^{-1}k^a + \mathrm{O}(c^{-3}))(c\tau + c^{-1}a + \mathrm{O}(c^{-3})) \\
&\quad + \left(\delta^a_b + c^{-2}\frac{k^a k_b}{2} + \mathrm{O}(c^{-4})\right)(\mathrm{e}^b + \mathrm{O}(c^{-2})) \\
&= k^a\tau + \mathrm{e}^a + \mathrm{O}(c^{-2}), && (5.43b) \\
c^{-1}\tilde{v} + \mathrm{O}(c^{-3}) &= \tilde{\mathrm{E}}_0 = (\Lambda^{-1})^A{}_0 \mathrm{E}_A \\
&= (\Lambda^{-1})^0{}_0 \mathrm{E}_0 + (\Lambda^{-1})^a{}_0 \mathrm{E}_a \\
&= \left(1 + c^{-2}\frac{|k|^2}{2} + \mathrm{O}(c^{-4})\right)(c^{-1}v + \mathrm{O}(c^{-3})) \\
&\quad + (-c^{-1}k^a + \mathrm{O}(c^{-3}))(\mathrm{e}_a + \mathrm{O}(c^{-2})) \\
&= c^{-1}(v - k^a \mathrm{e}_a) + \mathrm{O}(c^{-3}), && (5.43c) \\
\tilde{\mathrm{e}}_a + \mathrm{O}(c^{-2}) &= \tilde{\mathrm{E}}_a = (\Lambda^{-1})^B{}_a \mathrm{E}_B \\
&= (\Lambda^{-1})^0{}_a \mathrm{E}_0 + (\Lambda^{-1})^b{}_a \mathrm{E}_b \\
&= (-c^{-1}k_a + \mathrm{O}(c^{-3}))(c^{-1}v + \mathrm{O}(c^{-3})) \\
&\quad + \left(\delta^b_a + c^{-2}\frac{k^b k_a}{2} + \mathrm{O}(c^{-4})\right)(\mathrm{e}_b + \mathrm{O}(c^{-2})) \\
&= \mathrm{e}_a + \mathrm{O}(c^{-2}). && (5.43d)
\end{aligned}$$

From this, we may read off the transformation behaviour

$$\begin{aligned}
(v, \mathrm{e}_a) &\to (v - k^a \mathrm{e}_a, \mathrm{e}_a) = (v, \mathrm{e}_a) \cdot (\mathbb{1}, k)^{-1}, && (5.44a) \\
(\tau, \mathrm{e}^a) &\to (\tau, \mathrm{e}^a + k^a \tau), && (5.44b) \\
a &\to a + k_a \mathrm{e}^a + \frac{1}{2}|k|^2\tau, && (5.44c)
\end{aligned}$$

which is the correct transformation behaviour for local Galilei frames, their dual frames, and the local representatives of Bargmann structures under local Galilei boosts, according to propositions 3.16 and 5.6 (i).

(iv) The local connection form $(\overset{\mathrm{L}}{\omega}{}^A{}_B)$ of $\overset{\mathrm{L}}{\boldsymbol{\omega}}$ with respect to the local orthonormal frame (E_A) takes values in the Lorentz algebra, i.e. we have

$$\overset{\mathrm{L}}{\omega}{}^0{}_0 = 0, \quad \overset{\mathrm{L}}{\omega}{}^a{}_0 = \delta^{ab}\overset{\mathrm{L}}{\omega}{}^0{}_b, \quad (\overset{\mathrm{L}}{\omega}{}^a{}_b) \in \mathfrak{so}(n). \tag{5.45}$$

According to the general behaviour of local connection forms under frame changes, $(\overset{\mathrm{L}}{\omega}{}^A{}_B)$ can be expressed in terms of the local connection form $(\widehat{\overset{\mathrm{L}}{\omega}}{}^A{}_B)$ of $\overset{\mathrm{L}}{\boldsymbol{\omega}}$ with respect to the frame (v, e_a) as

$$\overset{\mathrm{L}}{\omega}{}^A{}_B = (\mathrm{X}^{-1})^A{}_C\, \widehat{\overset{\mathrm{L}}{\omega}}{}^C{}_D\, \mathrm{X}^D{}_B + (\mathrm{X}^{-1})^A{}_C\, \mathrm{d}\mathrm{X}^C{}_B \tag{5.46a}$$

with

$$\mathrm{E}_A = \mathrm{X}^B{}_A \mathrm{e}_B\,. \tag{5.46b}$$

From the frame and dual frame expansions, we obtain the frame change matrix and its inverse as

$$\mathrm{X}^t{}_0 = \tau(\mathrm{E}_0) = c^{-1} + \mathrm{O}(c^{-3}), \qquad \mathrm{X}^a{}_0 = \mathrm{e}^a(\mathrm{E}_0) = \mathrm{O}(c^{-3}), \tag{5.47a}$$
$$\mathrm{X}^t{}_a = \tau(\mathrm{E}_a) = \mathrm{O}(c^{-2}), \qquad \mathrm{X}^a{}_b = \mathrm{e}^a(\mathrm{E}_b) = \delta^a_b + \mathrm{O}(c^{-2}), \tag{5.47b}$$
$$(\mathrm{X}^{-1})^0{}_t = \mathrm{E}^0(v) = c + \mathrm{O}(c^{-1}), \qquad (\mathrm{X}^{-1})^a{}_t = \mathrm{E}^a(v) = \mathrm{O}(c^{-2}), \tag{5.47c}$$
$$(\mathrm{X}^{-1})^0{}_a = \mathrm{E}^0(\mathrm{e}_a) = \mathrm{O}(c^{-1}), \qquad (\mathrm{X}^{-1})^a{}_b = \mathrm{E}^a(\mathrm{e}_b) = \delta^a_b + \mathrm{O}(c^{-2}). \tag{5.47d}$$

Together with the assumption that the $\widehat{\overset{\mathrm{L}}{\omega}}{}^A{}_B$ have regular formal $c \to \infty$ limits, which means that they are of the order c^0, we thus obtain

$$\overset{\mathrm{L}}{\omega}{}^a{}_0 = \underbrace{(\mathrm{X}^{-1})^a{}_C\, \widehat{\overset{\mathrm{L}}{\omega}}{}^C{}_D\, \mathrm{X}^D{}_0}_{=c^{-1}\widehat{\overset{\mathrm{L}}{\omega}}{}^a{}_t + \mathrm{O}(c^{-3})} + (\mathrm{X}^{-1})^a{}_C\, \underbrace{\mathrm{d}\mathrm{X}^C{}_0}_{=\mathrm{O}(c^{-3})} = c^{-1}\widehat{\overset{\mathrm{L}}{\omega}}{}^a{}_t + \mathrm{O}(c^{-3}), \tag{5.48a}$$

$$\overset{\mathrm{L}}{\omega}{}^a{}_b = \underbrace{(\mathrm{X}^{-1})^a{}_C\, \widehat{\overset{\mathrm{L}}{\omega}}{}^C{}_D\, \mathrm{X}^D{}_b}_{=\widehat{\overset{\mathrm{L}}{\omega}}{}^a{}_b + \mathrm{O}(c^{-2})} + (\mathrm{X}^{-1})^a{}_C\, \underbrace{\mathrm{d}\mathrm{X}^C{}_b}_{=\mathrm{O}(c^{-2})} = \widehat{\overset{\mathrm{L}}{\omega}}{}^a{}_b + \mathrm{O}(c^{-2}), \tag{5.48b}$$

which proves an expansion as in (5.39).

A direct calculation as in the proof of (iii) shows that the local forms $(\omega^a{}_b, \varpi^a)$ transform under local Galilei boosts of the frame as the local connection form of a Galilei connection would. One easily sees that the same is true for spatial rotations since those are c-independent and thus rotations of (E_a) directly give rotations of (e_a). Therefore, we obtain a Galilei connection $\boldsymbol{\omega}$.

Finally, from the expansions of the dual frame (5.36) and the connection form (5.39) we obtain the torsion of $\overset{\mathrm{L}}{\omega}$ using Cartan's first structure equation (proposition 3.14 (ii)) as

$$\begin{aligned}\overset{\mathrm{L}}{T}{}^0 &= \mathrm{dE}^0 + \overset{\mathrm{L}}{\omega}{}^0{}_A \wedge \mathrm{E}^A \\ &= c\mathrm{d}\tau + c^{-1}\mathrm{d}a + c^{-1}\varpi_a \wedge \mathrm{e}^a + \mathrm{O}(c^{-3}) \\ &= c\mathrm{d}\tau + c^{-1}f + \mathrm{O}(c^{-3}), \end{aligned} \tag{5.49a}$$

$$\begin{aligned}\overset{\mathrm{L}}{T}{}^a &= \mathrm{dE}^a + \overset{\mathrm{L}}{\omega}{}^a{}_B \wedge \mathrm{E}^B \\ &= \mathrm{de}^a + \varpi^a\tau + \omega^a{}_b \wedge \mathrm{e}^b + \mathrm{O}(c^{-2}) \\ &= T^a + \mathrm{O}(c^{-2}). \end{aligned} \tag{5.49b}$$

□

Construction 5.14. We now want to analyse how the formal $c \to \infty$ limit from Lorentzian manifolds to Galilei structures interacts with (active) transformations by diffeomorphisms. With notations as in theorem 5.13, we let $\varphi\colon M \to M$ be a diffeomorphism, and consider its action on all the Lorentzian objects by pushforward, i.e.

$$g \to \varphi_* g, \quad \mathrm{E}_A \to \varphi_* \mathrm{E}_A, \quad \overset{\mathrm{L}}{\omega} \to \varphi_* \overset{\mathrm{L}}{\omega}, \tag{5.50a}$$

where the action of φ on the connection $\overset{\mathrm{L}}{\omega}$ is in the natural way, which may be either described by pushforward of local connection forms on $U \subset M$ or by pushforward of the global connection form on the linear frame bundle $F(M)$ by the natural lift of φ to $F(M)$. By duality, it is clear that the dual frame transforms as

$$\mathrm{E}^A \to \varphi_* \mathrm{E}^A. \tag{5.50b}$$

Now directly inserting the formal expansions in powers of c^{-1}, we see that the 'limiting' objects for our Galilei manifold also transform by pushforward, i.e. as[8]

$$\tau \to \varphi_*\tau, \qquad h \to \varphi_* h, \qquad \mathrm{e}^a \to \varphi_*\mathrm{e}^a, \tag{5.51a}$$

$$v \to \varphi_* v, \qquad \mathrm{e}_a \to \varphi_*\mathrm{e}_a, \tag{5.51b}$$

$$a \to \varphi_* a, \qquad \boldsymbol{\omega} \to \varphi_*\boldsymbol{\omega}. \tag{5.51c}$$

[8] One easily checks that if a is the local representative of a Bargmann structure on a Galilei manifold (M, τ, h) with respect to the Galilei frame (e_A), then for any diffeomorphism $\varphi\colon M \to N$ the form $\varphi_* a$ will be the local representative of a Bargmann structure on the Galilei manifold $(N, \varphi_*\tau, \varphi_* h)$ with respect to the Galilei frame $(\varphi_*\mathrm{e}_A)$, so we can really transport Bargmann structures by pushforward by pushing forward their local representatives.

Now, instead of considering transformations by proper diffeomorphisms of M, we want to transform the fields by 'c-dependent diffeomorphisms'. Of course, strictly speaking this does not make mathematical sense, since coordinate representations of diffeomorphisms are $\mathbb{R}^n$-valued, and thus cannot be formal power series in c^{-1}. However, when considering a family of diffeomorphisms φ_ε smoothly depending on a proper small parameter ε with $\varphi_0 = \mathrm{id}_M$, its pushforward action on any natural geometric object A may be expanded using the Lie derivative as

$$(\varphi_\varepsilon)_* A = A - \varepsilon \mathcal{L}_X A + \mathrm{O}(\varepsilon^2), \tag{5.52}$$

where X is the vector field generating of the family of diffeomorphisms. Therefore, the way to properly implement 'pushforward by c-dependent diffeomorphisms close to the identity' is to consider the action

$$A \to A - c^{-2} \mathcal{L}_X A + \mathrm{O}(c^{-4}) \tag{5.53a}$$

for a vector field X, which is thought of as being the leading order expansion of action with the pushforward of the diffeomorphism $\exp(c^{-2}X)$.[9] Now applying this action to some object A expanded as

$$A = c^{-k}(A_{(0)} + c^{-2} A_{(2)} + \mathrm{O}(c^{-4})) \,, \tag{5.53b}$$

we see that the leading coefficient stays invariant, and only starting at the next order coefficient we obtain a non-trivial transformation, namely

$$A \to c^{-k} \left(A_{(0)} + c^{-2}(A_{(2)} - \mathcal{L}_X A_{(0)}) + \mathrm{O}(c^{-4}) \right). \tag{5.53c}$$

Applying this to all our Lorentzian objects, for the 'limiting' Galilei-manifold objects the only non-trivial transformation we obtain is

$$a \to a - \mathcal{L}_X \tau = a - \mathrm{d}\tau(X, \cdot) - \mathrm{d}(\tau(X)), \tag{5.54}$$

where we used Cartan's 'magic formula'.

In the case of absolute time, i.e. $\mathrm{d}\tau = 0$, this amounts to

$$a \to a - \mathrm{d}\tau(X, \cdot) - \mathrm{d}(\tau(X)) = a - \mathrm{d}(\tau(X)), \tag{5.55}$$

i.e. to an $\mathbb{R}$ gauge transformation of the Bargmann structure as in remark 5.10 with transformation parameter $\chi = -\tau(X)$. So in the case $\mathrm{d}\tau = 0$, *we can obtain all 'natural symmetries' of the framework of Galilei manifolds with Bargmann structure – diffeomorphisms,*

[9] Considering $c^{-1}X$ instead would spoil the expansions assumed in theorem 5.13.

local Galilei transformations, and $\mathbb{R}$ gauge transformations of the Bargmann structure – from the action of 'c-dependent' diffeomorphisms and local Lorentz transformations on Lorentzian objects.

In the case $\mathrm{d}\tau \neq 0$, the transformation (5.54) of Bargmann structures arising from 'c-dependent infinitesimal diffeomorphisms' does not have a similar interpretation as directly rooted in the structure of the Bargmann group as in the case $\mathrm{d}\tau = 0$.

6. Outlook

Here, we will very briefly comment on some further topics in Newton–Cartan gravity which were not covered in the lectures or exercises.

The first topic we want to mention is the method of obtaining Newton–Cartan gravity from so-called *null reduction* of higher-dimensional Lorentzian geometry, which was first introduced in [Duv+85] and further developed in [JN95] (and many later works). One considers Lorentzian geometry in $n+2$ dimensions, with a null (i.e. lightlike) Killing vector field ξ. Assuming that the quotient of the Lorentzian manifold by the flow of ξ, i.e. the space of flow lines, be a manifold, this quotient is a Galilei manifold with a Bargmann structure in a natural way. Here all natural symmetries of the framework of Galilei manifolds with Bargmann structure – i.e. diffeomorphisms, local Galilei transformations, and $\mathbb{R}$ gauge transformations of the Bargmann structure – arise from Lorentzian symmetries, also in the case $\mathrm{d}\tau \neq 0$ (differently to the formal Newtonian limit of Lorentzian geometry, see construction 5.14). Similarly to the case of the formal $c \to \infty$ limit, matter coupling on the Galilei spacetime may be obtained from matter coupling to the Lorentzian spacetime. Assuming the Einstein equations for the Lorentzian spacetime leads to $\mathrm{d}\tau = 0$, and yields the Newton–Cartan field equation for the Galilei manifold.

As a second topic, we want to comment on so-called *torsional Newton–Cartan gravity* (TNC gravity), first introduced in the context of holography in [Chr+14a; Chr+14b]. Here one considers Galilei manifolds with $\mathrm{d}\tau \neq 0$, i.e. with non-vanishing temporal torsion for Galilei connections – hence the name. Such manifolds arise for example by null reduction of general Lorentzian manifolds; they are widely considered in applications of Galilei geometry in condensed matter theory and string theory. An important special case is so-called 'twistless torsional Newton–Cartan gravity' (TTNC gravity): here one assumes $\tau \wedge \mathrm{d}\tau = 0$, such that according to the Frobenius theorem hypersurfaces integrating the spacelike distribution still exist. Physically this means that in such Galilei spacetimes, one still has an absolute notion of simultaneity, but no absolute notion of time (if $\mathrm{d}\tau \neq 0$), which may be taken as modelling 'gravitational time dilation' in a (generalised) Newtonian context – see, however, the caveat on symmetries below.

Finally, there is a variant of TNC gravity which by goes by the name *TNC type II* (with the original version being termed *type I* in retrospect), first developed in [HHO19a; HHO19b; HHO20]. As we have seen above in construction 5.14, in the case $\mathrm{d}\tau \neq 0$ a formal expansion in c^{-1} of Lorentzian geometry with its natural symmetries does not lead to all natural symmetries of the framework of Bargmann structures on Galilei manifolds. To understand also the new symmetry (5.54) of the local one-forms a that arises from the formal expansion of 'c-dependent diffeomorphisms' as some local 'gauge transformation', one needs to modify the local symmetry algebra: instead of the Bargmann algebra, one needs to consider a different, non-central extension of the Galilei algebra, which actually can be obtained by a so-called *Lie algebra expansion* of the Poincaré algebra. This geometric framework is TNC type II geometry. It allows for the consistent description of (generalised) 'Newtonian limits' of Lorentzian spacetimes with $\mathrm{d}\tau \neq 0$, i.e. including 'time dilation', and enables the formulation of an action principle for Newtonian gravity (including first-order post-Newtonian corrections). One can also employ this framework to study the post-Newtonian expansion of (locally) Poincaré-relativistic theories of gravity (i.e. GR or its modifications) in a coordinate-free, geometric way.

A. Semidirect products of Lie groups

Definition A.1. Let H, N be Lie groups and $\rho\colon H \to \mathrm{Aut}(N)$ a Lie group homomorphism[1] (here $\mathrm{Aut}(N)$ is the group of Lie group automorphisms of N, i.e. isomorphisms $N \to N$). The *semidirect product* of H and N with respect to ρ, denoted $H \ltimes_\rho N$, is the Lie group whose underlying manifold is the product $H \times N$ and whose group operation is

$$(h,n)(\tilde{h},\tilde{n}) = (h\tilde{h}, n\rho_h(\tilde{n})), \tag{A.1}$$

i.e. in the group operation 'H acts on N via ρ'.

That this operation is really associative is guaranteed by ρ being a homomorphism and taking values in automorphisms. The neutral element of $H \ltimes_\rho N$ is $(\mathrm{e}_H, \mathrm{e}_N)$, and inverses are given by

$$(h,n)^{-1} = (h^{-1}, \rho_{h^{-1}}(n^{-1})) \tag{A.2}$$

(if you have never encountered semidirect products before, check all this as an exercise!). If the homomorphism by which H acts on N is clear from context, we will omit it from the notation.

Both H and N embed as subgroups into $H \ltimes N$ via the maps

$$H \ni h \mapsto (h, \mathrm{e}_N) \in H \ltimes N, \tag{A.3a}$$

$$N \ni n \mapsto (\mathrm{e}_H, n) \in H \ltimes N. \tag{A.3b}$$

N is a normal subgroup of $H \ltimes N$, with the quotient being canonically isomorphic to H. This means we have a short exact sequence

$$1 \to N \to H \ltimes N \to H \to 1. \tag{A.4}$$

In general, H as a subgroup is not normal.

An analogous construction exists for Lie algebras:

[1] In general, if N is not connected and the group of its connected components is not finitely generated, $\mathrm{Aut}(N)$ is not a Lie group. However, in fact we just need ρ to be smooth in the sense that the joint map $H \times N \ni (h,n) \mapsto \rho_h(n) \in N$ is smooth, which makes sense in any case.

Definition A.2. Let $\mathfrak{h}, \mathfrak{n}$ be Lie algebras and $\hat{\rho}\colon \mathfrak{h} \to \mathrm{Der}(\mathfrak{n})$ a Lie algebra homomorphism (here $\mathrm{Der}(\mathfrak{n})$ is the Lie algebra of derivations on $\mathfrak{n}$, i.e. linear maps $f\colon \mathfrak{n} \to \mathfrak{n}$ satisfying a 'Leibniz rule' with respect to the Lie bracket, $f([Y,\tilde{Y}]) = [f(Y),\tilde{Y}] + [Y, f(\tilde{Y})]$). The *semidirect sum* of $\mathfrak{h}$ and $\mathfrak{n}$ with respect to $\hat{\rho}$, denoted $\mathfrak{h} \oplus_{\hat{\rho}} \mathfrak{n}$, is the Lie algebra whose underlying vector space is the direct sum $\mathfrak{h} \oplus \mathfrak{n}$ and whose Lie bracket is

$$[(X,Y),(\tilde{X},\tilde{Y})] = ([X,\tilde{X}], [Y,\tilde{Y}] + \hat{\rho}_X(\tilde{Y}) - \hat{\rho}_{\tilde{X}}(Y)), \tag{A.5}$$

i.e. in the Lie bracket '$\mathfrak{h}$ acts on $\mathfrak{n}$ via $\hat{\rho}$'.[2]

That this bracket really satisfies the Jacobi identity is guaranteed by $\hat{\rho}$ being a homomorphism and taking values in derivations (if you have never encountered semidirect sums before, check this as an exercise!). As for semidirect products of groups, if the homomorphism is clear from context, we will omit it from the notation.

Both $\mathfrak{h}$ and $\mathfrak{n}$ embed as Lie subalgebras into $\mathfrak{h} \oplus \mathfrak{n}$ via the maps

$$\mathfrak{h} \ni X \mapsto (X,0) \in \mathfrak{h} \oplus \mathfrak{n}, \tag{A.6a}$$
$$\mathfrak{n} \ni Y \mapsto (0,Y) \in \mathfrak{h} \oplus \mathfrak{n}. \tag{A.6b}$$

$\mathfrak{n}$ is an ideal of $\mathfrak{h} \oplus \mathfrak{n}$, with the quotient being canonically isomorphic to $\mathfrak{h}$. This means we have a short exact sequence

$$0 \to \mathfrak{n} \to \mathfrak{h} \oplus \mathfrak{n} \to \mathfrak{h} \to 0. \tag{A.7}$$

In general, $\mathfrak{h}$ as a subalgebra is no ideal.

Semidirect products of Lie groups and semidirect sums of Lie algebras are closely related. To specify this, we need the following.

Construction A.3. Let H, N be Lie groups and $\rho\colon H \to \mathrm{Aut}(N)$ a Lie group homomorphism. For each $h \in H$, we consider the differential of $\rho_h\colon N \to N$ at the neutral element $\mathrm{e}_N \in N$, which we denote by

$$\dot{\rho}_h := \mathrm{D}(\rho_h)\big|_{\mathrm{e}_N} : T_{\mathrm{e}_N}N \to T_{\mathrm{e}_N}N. \tag{A.8}$$

Since ρ_N is a Lie group automorphism, this differential is a Lie algebra automorphism $\dot{\rho}_h \in \mathrm{Aut}(\mathfrak{n})$. By the chain rule, the fact that ρ is a Lie group homomorphism translates into

$$\dot{\rho}\colon H \to \mathrm{Aut}(\mathfrak{n}), h \mapsto \dot{\rho}_h = \mathrm{D}(\rho_h)\big|_{\mathrm{e}_N} \tag{A.9}$$

[2]Note that the notation with $\oplus$ is non-standard. Most literature simply uses the direct sum symbol, but we want to take account of the non-trivial Lie bracket structure in the notation, and have it reflect the 'direction' of the action ('from left to right'), similar to the standard $\ltimes$ notation for semidirect products.

being a Lie group homomorphism.

Now of *this* homomorphism, we again take the differential at the neutral element $\mathrm{e}_H \in H$, which is a Lie algebra homomorphism

$$\dot{\rho}' := \mathrm{D}\dot{\rho}\big|_{\mathrm{e}_H} : \mathfrak{h} \to \mathrm{Lie}(\mathrm{Aut}(\mathfrak{n})). \tag{A.10}$$

But the Lie algebra of $\mathrm{Aut}(\mathfrak{n})$ consists of derivations on $\mathfrak{n}$ (if you want, check this as an exercise), so we have seen how a Lie group homomorphism $\rho\colon H \to \mathrm{Aut}(N)$ induces a Lie algebra homomorphism

$$\dot{\rho}'\colon \mathfrak{h} \to \mathrm{Der}(\mathfrak{n}). \tag{A.11}$$

Proposition A.4. *Let H, N be Lie groups and $\rho\colon H \to \mathrm{Aut}(N)$ a Lie group homomorphism.*

(i) The Lie algebra of the semidirect product $H \ltimes_\rho N$ is the semidirect sum $\mathfrak{h} \oplus_{\dot{\rho}'} \mathfrak{n}$, where $\dot{\rho}'\colon \mathfrak{h} \to \mathrm{Der}(\mathfrak{n})$ is the Lie algebra homomorphism induced by ρ (according to construction A.3).

(ii) The adjoint representation of $H \ltimes_\rho N$ is given by

$$\mathrm{Ad}_{(h,n)}(X, Y) = (\mathrm{Ad}_h(X), \mathrm{Ad}_n(\dot{\rho}_h(Y)) + \sigma_n(\mathrm{Ad}_h(X))) \tag{A.12}$$

for $(h, n) \in H \ltimes_\rho N$ and $(X, Y) \in \mathfrak{h} \oplus_{\dot{\rho}'} \mathfrak{n}$, where $\dot{\rho}\colon H \to \mathrm{Aut}(\mathfrak{n})$ is the homomorphism induced by ρ, and $\sigma_n\colon \mathfrak{h} \to \mathfrak{n}$ is the differential of the map $H \to N, h \mapsto n\rho_h(n^{-1})$ at the neutral element e_H.

Proof. As a vector space, we know that the Lie algebra of the semidirect product is $\mathrm{Lie}(H \ltimes_\rho N) = T_{(\mathrm{e}_H,\mathrm{e}_N)}(H \ltimes_\rho N) = T_{\mathrm{e}_H}H \oplus T_{\mathrm{e}_N}N = \mathfrak{h} \oplus \mathfrak{n}$. First, we will compute the adjoint representation. The conjugation map by $(h, n) \in (H \ltimes_\rho N)$ is given by

$$\begin{aligned}
\alpha_{(h,n)}(\tilde{h}, \tilde{n}) &= (h, n) \cdot (\tilde{h}, \tilde{n}) \cdot (h, n)^{-1} \\
&= (h\tilde{h}, n\rho_h(\tilde{n})) \cdot (h^{-1}, \rho_{h^{-1}}(n^{-1})) \\
&= \left(h\tilde{h}h^{-1}, n\rho_h(\tilde{n})\rho_{h\tilde{h}}(\rho_{h^{-1}}(n^{-1}))\right) \\
&= \left(\alpha_h(\tilde{h}), n\rho_h(\tilde{n})\rho_{\alpha_h(\tilde{h})}(n^{-1})\right).
\end{aligned} \tag{A.13}$$

Now, in order to compute the adjoint representation

$$\mathrm{Ad}_{(h,n)} := \mathrm{D}(\alpha_{(h,n)})\Big|_{(\mathrm{e}_H,\mathrm{e}_N)} : \mathfrak{h} \oplus \mathfrak{n} \to \mathfrak{h} \oplus \mathfrak{n}, \tag{A.14}$$

consider $(X, Y) \in \mathfrak{h} \oplus \mathfrak{n}$ and let $\tilde{h}(t), \tilde{n}(t)$ be curves in H and N respectively such that $\dot{\tilde{h}}(0) = X, \dot{\tilde{n}}(0) = Y$. We then may compute

$$\begin{aligned}
\mathrm{Ad}_{(h,n)}(X, Y) &= \frac{\mathrm{d}}{\mathrm{d}t}\alpha_{(h,n)}(\tilde{h}(t), \tilde{n}(t))\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\alpha_{(h,n)}(\tilde{h}(t), \mathrm{e}_N)\Big|_{t=0} + \frac{\mathrm{d}}{\mathrm{d}t}\alpha_{(h,n)}(\mathrm{e}_H, \tilde{n}(t))\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\alpha_h(\tilde{h}(t)), n\rho_{\alpha_h(\tilde{h}(t))}(n^{-1})\right)\Big|_{t=0} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}_H, n\rho_h(\tilde{n}(t))n^{-1}\right)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\alpha_h(\tilde{h}(t)), \beta_n(\alpha_h(\tilde{h}(t)))\right)\Big|_{t=0} + \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}_H, \alpha_n(\rho_h(\tilde{n}(t))))\Big|_{t=0}, \qquad \text{(A.15)}
\end{aligned}$$

where we introduced the notation $\beta_n(\hat{h}) := n\rho_{\hat{h}}(n^{-1})$. Denoting the differential of this map by $\sigma_n := \mathrm{D}\beta_n|_{\mathrm{e}_H} : \mathfrak{h} \to \mathfrak{n}$ as in the statement of the proposition, we obtain

$$\begin{aligned}
\mathrm{Ad}_{(h,n)}(X, Y) &= \left(\mathrm{Ad}_h(\dot{\tilde{h}}(0)), \sigma_n(\mathrm{Ad}_h(\dot{\tilde{h}}(0)))\right) + (0, \mathrm{Ad}_n(\dot{\rho}_h(\dot{\tilde{n}}(0)))) \\
&= (\mathrm{Ad}_h(X), \mathrm{Ad}_n(\dot{\rho}_h(Y)) + \sigma_n(\mathrm{Ad}_h(X)))\,. \qquad \text{(A.16)}
\end{aligned}$$

To prove that the Lie algebra is the semidirect sum, we may now use this result: we use the fact that $[(X, Y), (\tilde{X}, \tilde{Y})] = \mathrm{ad}_{(X,Y)}(\tilde{X}, \tilde{Y})$, where the adjoint representation of the Lie algebra may be obtained from that of the Lie group by differentiating,

$$\mathrm{ad} = \mathrm{D}(\mathrm{Ad})|_{(\mathrm{e}_H, \mathrm{e}_N)}\,. \qquad \text{(A.17)}$$

Details are left as an exercise.[3] □

[3] In order to avoid the necessity of computing $\frac{\mathrm{d}}{\mathrm{d}t}\sigma_{n(t)}(\tilde{X})$ in this calculation, which may become quite cumbersome, it is easiest to compute only Lie brackets of the form $[(X, 0), (\tilde{X}, 0)], [(0, Y), (0, \tilde{Y})], [(X, 0), (0, Y)]$ by differentiating Ad, and use those to express a general bracket.

Of course, one can also compute the Lie bracket of a semidirect product group via the consideration of left invariant vector fields, but the way via the adjoint representation is somewhat easier.

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